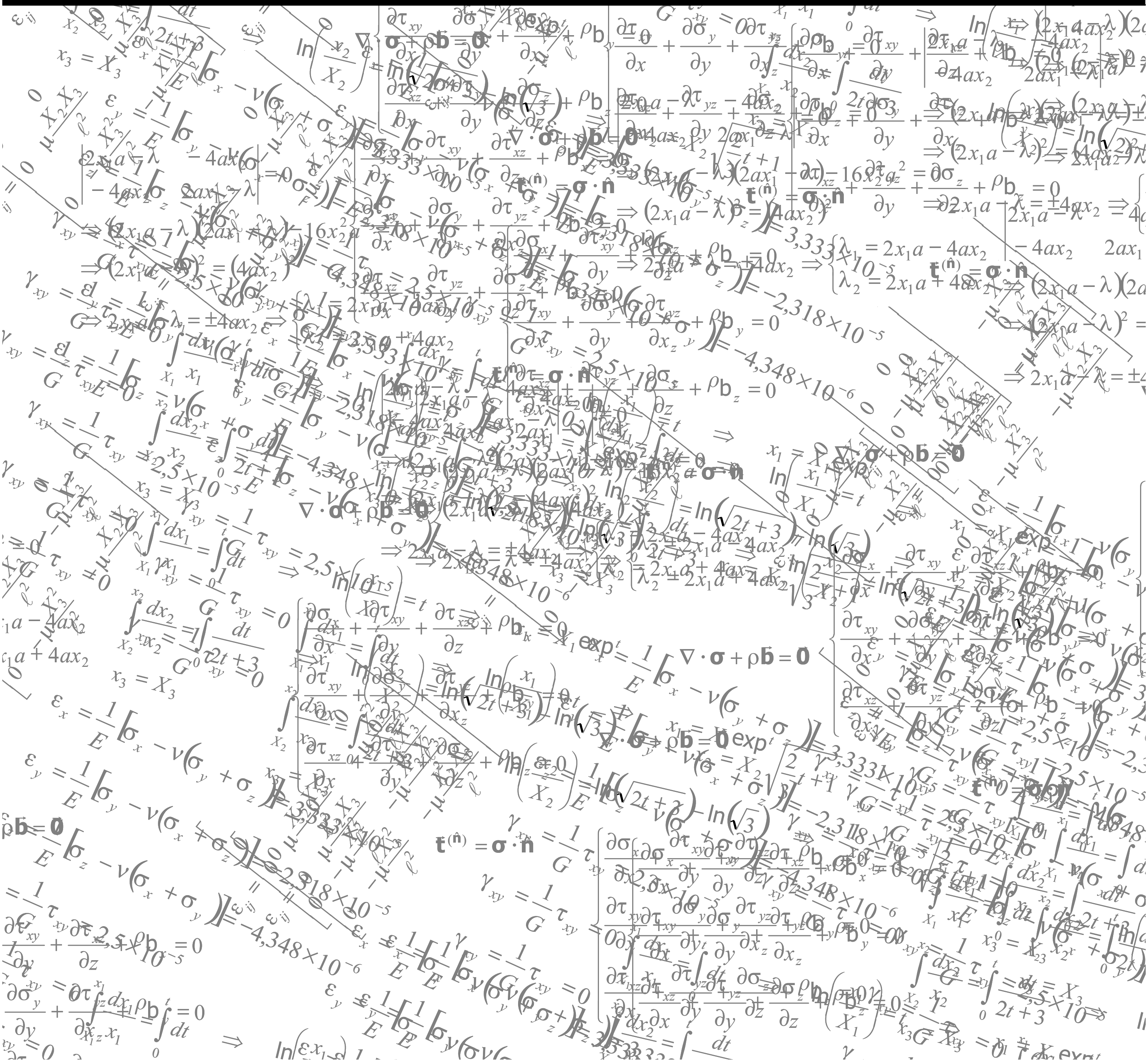
SOLVING PROBLEMS BY MEANS OF CONTINUUM MECHANICS

Eduardo W. V. Chaves



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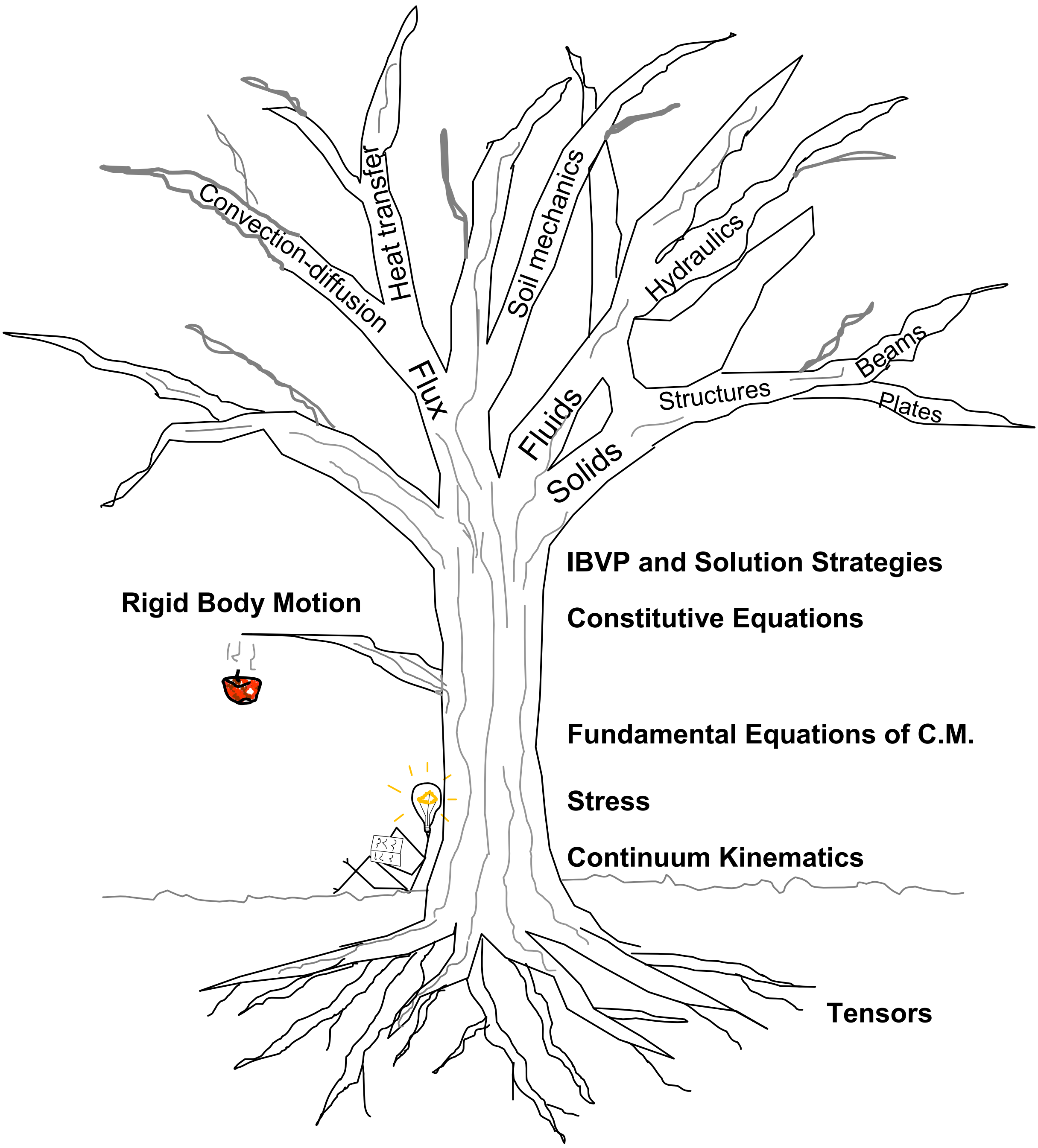
Presentation



Contents

Abbreviations

IBVP	Initial Boundary Value Problem
BVP	Boundary Value Problem
FEM	Finite Element Method
BEM	Boundary Element Method
FDM	Finite Difference Method
C.M.	Continuum Mechanics
iff	if and only if

Latin

i.e.	*id est*	that is
et al.	*et alii*	and the others
e.g.	*exempli gratia*	for example
etc.	*et cetera*	and so on
Q.E.D.	*Quod Erat Demonstrandum*	which had to be demonstrated
v., vs.	*versus*	versus
viz.	*vidilicet*	namely

Operators and Symbols

$\langle\bullet\rangle=\dfrac{\lvert\bullet\rvert+\bullet}{2}$	Macaulay bracket
$\lVert\bullet\rVert$	Euclidian norm of $\bullet$
$\mathrm{Tr}(\bullet)$	trace of $(\bullet)$
$(\bullet)^T$	transpose of $(\bullet)$
$(\bullet)^{-1}$	inverse of $(\bullet)$
$(\bullet)^{-T}$	inverse of the transpose of $(\bullet)$
$(\bullet)^{sym}$	symmetric part of $(\bullet)$
$(\bullet)^{skew}$	antisymmetric (skew-symmetric) part of $(\bullet)$
$(\bullet)^{sph}$	spherical part of $(\bullet)$
$(\bullet)^{dev}$	deviatoric part of $(\bullet)$
$\lVert\bullet\rVert$	module of $\bullet$
$[[\bullet]]$	jump of $\bullet$
$\cdot$	scalar product
$\det(\bullet)\equiv\lvert\bullet\rvert$	determinant of $(\bullet)$
$\dfrac{D\bullet}{Dt}\equiv\dot{\bullet}$	material time derivative of $(\bullet)$
$\mathrm{cof}(\bullet)$	cofactor of $\bullet$;
$\mathrm{adj}(\bullet)$	adjugate of $(\bullet)$
$\mathrm{Tr}(\bullet)$	trace of $(\bullet)$
$:$	double scalar product (or double contraction or double dot product)
∇^2	Scalar differential operator
$\otimes$	tensorial product
$\nabla\bullet\equiv\mathrm{grad}(\bullet)$	gradient of $\bullet$
$\nabla\cdot\bullet\equiv\mathrm{div}(\bullet)$	divergence of $\bullet$
$\wedge$	vector product (or cross product)
$I_\bullet, II_\bullet, III_\bullet$	first, second and third principal invariants of the tensor $\bullet$
$\vec{\bullet}$	vector
$\hat{\bullet}$	unit vector
$\mathbf{1}$	Second-order unit tensor
$\mathbb{I}$	fourth-order unit tensor
$\mathbb{I}^{sym}\equiv\mathbf{I}$	symmetric fourth-order unit tensor

SI-Units

length	m **- meter**	electric current	A **- ampere**
mass	kg **- kilogram**	amount of substance	mol **- mole**
time	s **- second**	luminous intensity	cd **- candela**
temperature	K **- kelvin**		

velocity	$\frac{m}{s}$	energy, work, heat	$J = Nm$ - Joules
acceleration	$\frac{m}{s^2}$	power	$\frac{J}{s} \equiv W$ watt
energy	$J = Nm$ - Joules		
force	N - Newton	permeability	m^2
pressure, stress	$Pa \equiv \frac{N}{m^2}$ - Pascal	dynamic viscosity	$Pa \times s$
frequency	$\frac{1}{s} \equiv Hz$ - Hertz	mass flux	$\frac{kg}{m^2 s}$
thermal conductivity	$\frac{W}{mK}$	energy flux	$\frac{J}{m^2 s}$
mass density	$\frac{kg}{m^3}$	energy density	$\frac{J}{m^3}$

Prefix	Symbol	10^n	Prefix	Symbol	10^n
pico	p	10^{-12}	kilo	k	10^3
nano	η	10^{-9}	Mega	M	10^6
micro	μ	10^{-6}	Giga	G	10^9
mili	m	10^{-3}	Tera	T	10^{12}
centi	c	10^{-2}			
deci	d	10			

Physical Constants

Newtonian constant of gravitation: $G = 6.67384 \times 10^{-11} \dfrac{m^3}{kg\, s^2}$

Speed of light in vacuum: $c = 299\,792\,458 \dfrac{m}{s} \approx 300\,000\,000 \dfrac{m}{s}$

Absolute zero (temperature): $T = 0K = -273.15^{\circ}C$

$\pi = 3.14159 \quad 26535 \quad 89793 \quad 23846$

Nomenclature

$\vec{A}(\vec{X},t) \equiv \vec{a}(\vec{X},t)$	Acceleration (reference configuration)	$\frac{m}{s^2}$
$\mathcal{A}$	Transformation matrix	
$\vec{a}(\vec{x},t)$	Acceleration (current configuration)	$\frac{m}{s^2}$
$\mathcal{B}_0$	Continuum medium in the reference configuration at $t=0$	
$\mathcal{B}$	Continuum medium in the current configuration at time t	
$\partial\mathcal{B}$	Boundary of $\mathcal{B}$	
$\vec{\mathbf{b}}(\vec{x},t)$	Body force (per unit mass)	$\frac{N}{m^3}$
$\boldsymbol{b}$	Left deformation Cauchy-Green tensor, Finger deformation tensor	
$\boldsymbol{B}$	Piola deformation tensor	
$\boldsymbol{B}$	Entropy created inside	$\frac{J}{s\,K}$
$\boldsymbol{b}$	Local entropy per unit mass per unit time	$\frac{J}{kg\,s\,K}$
$\mathbb{C}^e$	Elasticity tensor	Pa
$[\mathcal{C}]$	Elasticity matrix (Voigt notation)	Pa
$\mathbb{C}^{in}$	Inelasticity tensor	Pa
$\boldsymbol{c}$	Cauchy deformation tensor	
C_v	Calor específico a volumen constante	
C_p	Calor específico a presión constante	
c	Cohesion	Pa
c_c	Solute concentration	$\frac{mol}{m^3}$
$\boldsymbol{C}$	Right deformation Cauchy-Green tensor	
D_V	Dilation	$\frac{m}{m}$
$\mathbf{D}$	Rate-of-Deformation tensor	
$d\vec{A}$	Area element vector in the reference configuration	m^2
$d\vec{a}$	Area element vector in the current configuration	m^2
dV	Volume element	m^3

$\boldsymbol{E}$	Green-Lagrange strain tensor, or Lagrangian finite strain tensor, or Green-St_Venant strain tensor	$\frac{m}{m}$
$\boldsymbol{e}$	Almansi strain tensor, or Eulerian finite strain tensor	$\frac{m}{m}$
E	Young's modulus, or elastic modulus	Pa
$\hat{\mathbf{e}}_i$	Cartesian basis in symbolic notation	
$\hat{\mathbf{i}},\hat{\mathbf{j}},\hat{\mathbf{k}}$	Cartesian basis	
$\boldsymbol{F}$	Deformation gradient (pseudo-tensor)	$\frac{m}{m}$
G	Shear modulus	Pa
$\boldsymbol{H}$	Biot strain tensor	
H	Total entropy	$\frac{J}{K}$
$\vec{\boldsymbol{H}}_O$	Angular momentum	$\frac{kgm^2}{s}=Js$
J	Jacobian determinant	$\frac{m^3}{m^3}$
$\boldsymbol{J}(\vec{X},t)$	Material displacement gradient tensor	$\frac{m}{m}$
$\boldsymbol{j}(\vec{x},t)$	Spatial displacement gradient tensor	$\frac{m}{m}$
$\vec{\mathbf{J}}$	Diffusion tensor	$\frac{mol}{m^2 s}$
$\mathbf{K}$	Thermal conductivity tensor	$\frac{W}{m\,K}=\frac{J}{s\,m\,K}$
$\mathcal{K}$	Kinetic energy	J
$\vec{\boldsymbol{L}}$	Linear momentum	$\frac{kg\,m}{s}$
$\boldsymbol{\ell}$	Spatial velocity gradient	$\frac{m}{s\,m}$
m	Mass	kg
$\mathbf{M}$	Mandel stress tensor	Pa
$\hat{\mathbf{n}}$	Outward unit normal to the boundary (current configuration)	
$\hat{\mathbf{N}}$	Outward unit normal to the boundary (reference configuration)	
$\vec{\mathbf{p}}$	Body force (per unit volume)	$\frac{N}{m^3}$
$\mathbf{P}$	First Piola-Kirchhoff stress tensor	Pa
p	Thermodynamic pressure	Pa
$\vec{\mathbf{q}}(\vec{x},t)$	Cauchy heat flux (non-convective vector)	$\frac{J}{m^2 s}$
$\mathbf{Q}$	Orthogonal tensor	
Q	Thermal work	J
$r(\vec{x},t)$	Radiant heat constant, or heat source (per unit mass)	$\frac{J}{kg\,s}$

$\mathbf{R}$	Orthogonal tensor of polar decomposition	
$\mathbf{S}$	Second Piola-Kirchhoff stress tensor	Pa
$\vec{s}$	Entropy flux	$\frac{J}{kg\ s\ m^2}$
$\mathbf{T}$	Biot stress tensor	Pa
$\vec{\mathbf{t}}^{(\hat{\mathbf{n}})}(\vec{x},t,\hat{\mathbf{n}})$	Traction vector (current configuration)	Pa
$\vec{\mathbf{t}}_0^{(\hat{\mathbf{N}})}$	Traction pseudo-vector (reference configuration)	Pa
$T(\vec{x},t)$	Temperature	K
t	Time	s
$t_0 \equiv t = 0$	Initial time	s
$\dot{U}$	Rate of change of the internal energy	$\frac{J}{s} = W$
u	Specific internal energy	$\frac{J}{kg}$
$\vec{\mathbf{u}}(\vec{x},t)$	Displacement vector (Eulerian)	m
$\vec{\mathbf{u}}(\vec{X},t)$	Displacement vector (Lagrangian)	m
$\mathbf{U}(\vec{X},t)$	Right stretch tensor, or Lagrangian stretch tensor, or material stretch tensor	
$\mathbf{V}(\vec{x},t)$	Left stretch tensor, or Eulerian stretch tensor, or spatial stretch tensor	
$\vec{V}(\vec{X},t) \equiv \vec{v}(\vec{X},t)$	Velocity (reference configuration)	$\frac{m}{s}$
$\vec{v}(\vec{x},t)$	Velocity (current configuration)	$\frac{m}{s}$
$\mathbf{W}$	Spin tensor, rate-of-rotation tensor, or vorticity tensor	$\frac{m}{ms} = \frac{rad}{s}$
$\mathrm{w}_{\mathrm{int}}$	Stress power	$\frac{J}{s} = W$
$\vec{X}$	Vector position (material coordinate)	m
$\vec{x}$	Vector position (spatial coordinate)	m
α	Coefficient of thermal expansion	$\frac{1}{K}$
δ_{ij}	Kronecker delta	
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	Principal strains (infinitesimal strain)	*dimensionless*
ε	Unit Extension	$\frac{m}{m}$ - *dimensionless*
ϵ_{ijk}	Permutation symbol, or Levi-Civita tensor components	
ε_V	Linear dilatation (volume ratio) (small deformation regime)	$\frac{m}{m}$ - *dimensionless*
$\boldsymbol{\varepsilon}$	Infinitesimal strain tensor	$\frac{m}{m}$ - *dimensionless*
η	Specific entropy	$\frac{J}{kg\ K}$
κ	Bulk modulus	Pa

κ	Thermal diffusivity	$\frac{m^2}{s}$
λ	Stretch	$\frac{m}{m}$
λ, μ	Lamé constants	Pa
ν	Poisson's ratio	
ρ_S	Solute mass density	$\frac{kg}{m^3}$
ρ_f	Fluid mass density	$\frac{kg}{m^3}$
$\rho_0(\vec{\boldsymbol{X}})$	Mass density (reference configuration)	$\frac{kg}{m^3}$
$\rho(\vec{\boldsymbol{x}}, t)$	Mass density (current configuration)	$\frac{kg}{m^3}$
$\frac{1}{\rho}$	Specific volume	$\frac{m^3}{kg}$
$\boldsymbol{\sigma}$	Cauchy stress tensor, or true stress tensor	Pa
$\vec{\boldsymbol{\sigma}}_N$	Normal traction vector	Pa
$\vec{\boldsymbol{\sigma}}_S$	Tangential traction vector	Pa
σ_m	Mean stress	Pa
$\sigma_1, \sigma_2, \sigma_3$	Principal stresses	Pa
$\vec{\boldsymbol{\sigma}}_{oct}$	Normal octahedral vector	Pa
$\vec{\boldsymbol{\tau}}_{oct}$	Tangential octahedral vector	Pa
$\tau_{\max}$	Maximum shear stress	Pa
$\boldsymbol{\tau}$	Kirchhoff stress tensor	Pa
ϕ	Angle of internal friction	
ψ	Helmholtz free energy, specific (per unit mass)	$\frac{J}{kg}$
Ψ	Helmholtz free energy (per unit volume)	$\frac{J}{m^3}$
$\Psi(\boldsymbol{\varepsilon}) = \Psi^e$	Strain energy density	$\frac{J}{m^3}$

Useful Formulas

Some Trigonometric Identities

$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$	$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)$
$\cos(\theta)\cos(\phi) = \frac{1}{2}[\cos(\theta+\phi) + \cos(\theta-\phi)]$	$\sin(\theta)\sin(\phi) = \frac{1}{2}[\cos(\theta-\phi) - \cos(\theta+\phi)]$
$\sin(\theta)\cos(\phi) = \frac{1}{2}[\sin(\theta+\phi) + \sin(\theta-\phi)]$	$\cos^2(\theta) = \frac{1}{2}[1+\cos(2\theta)]$
$\sin^2(\theta) = \frac{1}{2}[1-\cos(2\theta)]$	$\cos(\theta) + \cos(\phi) = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$
$\cos(\theta) - \cos(\phi) = 2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\phi-\theta}{2}\right)$	$\sin(\theta) \pm \sin(\phi) = 2\sin\left(\frac{\theta\pm\phi}{2}\right)\cos\left(\frac{\theta\mp\phi}{2}\right)$
$\cos^2(\theta) + \sin^2(\theta) = 1$	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
$\sec(\theta) = \frac{1}{\cos(\theta)}$	$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$
$\sec^2(\theta) + \tan^2(\theta) = 1$	

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \quad ; \quad \lim_{x\to 0}\frac{1-\cos(x)}{x} = 0$$

List of trigonometric identity
http://en.wikipedia.org/wiki/Trigonometric_identity

Some Series Expansions

$$f(x)=f(a)+\frac{\partial f}{\partial x}(x-a)+\frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(x-a)^2+\frac{1}{3!}\frac{\partial^3 f}{\partial x^3}(x-a)^3+\cdots \quad \text{(Taylor's series)}$$

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\cdots \quad ; \quad (\|x\|<1) \quad \text{(binomial series)}$$

$$\exp^x=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots$$

$$\mathrm{Ln}(1+x)=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots$$

$$\cos(x)=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots$$

$$\sin(x)=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots$$

$$\cosh(x)=1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots$$

$$\sinh(x)=x+\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots$$

$$\tan(x)=x+\frac{1}{3}x^3+\frac{1}{15}x^5+\cdots\left(\|x\|<\frac{\pi}{2}\right)$$

Some Derivatives

$$\frac{d}{dx}(\exp^x)=\exp^x \quad ; \quad \frac{d}{dx}(a^x)=\mathrm{Ln}(a)a^x \quad ; \quad \frac{d}{dx}[\mathrm{Ln}(x)]=\frac{1}{x} \quad ; \quad \frac{d}{dx}[\log_a(x)]=\frac{1}{x\mathrm{Ln}(a)}$$

$$\frac{d}{dx}[\mathrm{Ln}(f(x))]=\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}$$

where $\exp$ stands for exponential and Ln for natural logarithm, where it fulfils:

$\mathrm{Ln}(\exp^x)=x$ and $\exp^{\mathrm{Ln}(x)}=x$

$$\frac{d}{dx}[\sin(x)]=\cos(x) \quad ; \quad \frac{d}{dx}[\cos(x)]=-\sin(x) \quad ; \quad \frac{d}{dx}[\tan(x)]=\sec^2(x)$$

$$\frac{d}{dx}[\arcsin(x)]=\frac{1}{\sqrt{1-x^2}} \quad ; \quad \frac{d}{dx}[\arccos(x)]=\frac{-1}{\sqrt{1-x^2}} \quad ; \quad \frac{d}{dx}[\arctan(x)]=\frac{1}{1+x^2}$$

List of derivatives
http://en.wikipedia.org/wiki/List_of_derivatives

Some Integrals

$$\int \exp^{x} dx = \exp^{x} \quad ; \quad \int \frac{\partial f(x)}{\partial x} \exp^{f(x)} dx = \exp^{f(x)}$$

$$\int \frac{1}{x} dx = \mathrm{Ln}(x) \quad ; \quad \int \mathrm{Ln}(x) dx = x\mathrm{Ln}(x) - x + C$$

where $\mathrm{e} \equiv \exp$ stands for exponential and Ln for natural logarithm.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\left\|\frac{u}{a}\right\|\right) + C$$

List of integrals
http://en.wikipedia.org/wiki/List_of_integrals

Some Function Solutions

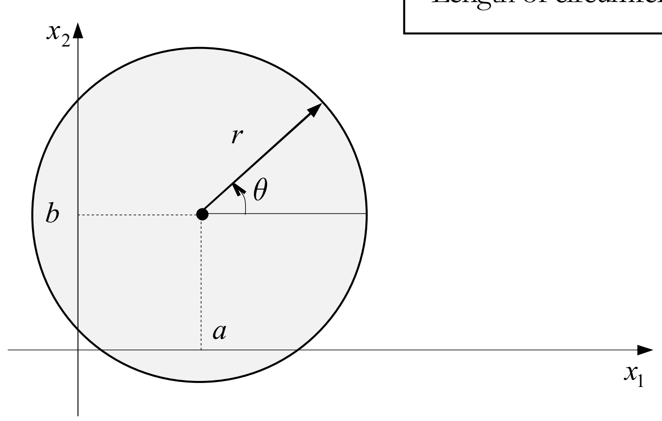
Quadratic function

$$ax^2 + bx + c = 0 \quad \xrightarrow{solution} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (a \neq 0)$$

Ruffini's rule
http://en.wikipedia.org/wiki/Ruffini%27s_rule

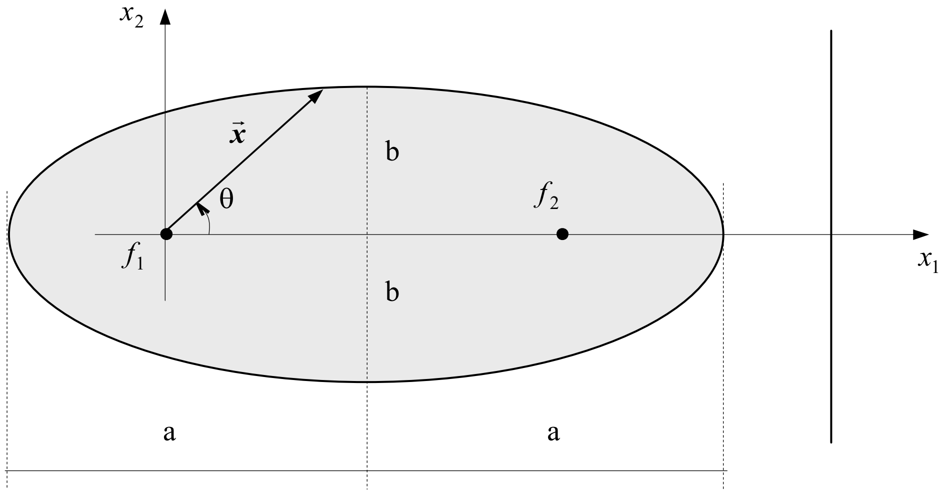
Expressions related to the circle:

Equation of the circle: $(x_1 - a)^2 + (x_2 - b)^2 \leq r^2$
Area enclosed by a circumference: $A = \pi r^2$
Length of circumference: $C = 2\pi r$



The relationship $ds = rd\theta$ holds, where ds is the infinitesimal arc length.

Expressions related to the ellipse:



Equation of the ellipse: $\|\vec{\boldsymbol{x}}\| = r = \dfrac{p}{1 + e\cos\theta}$

Eccentricity: $e = \sqrt{\dfrac{\mathrm{a}^2 - \mathrm{b}^2}{\mathrm{a}^2}}$; $0 < e < 1$, where $\mathrm{a}^2 = \dfrac{p^2}{(1 - e^2)^2}$ holds.

Area enclosed by an ellipse: $A = \pi \mathrm{ab}$.