Annex D

Introduction to Linear Viscoelasticity

D.1 Introduction

To understand viscoelastic behavior, we can carry out a simple experiment. For example, we can take a gum (used) and stretch it in such a way that most of the gum is concentrated at one end. Then, we place it in a vertical position so that the only force acting on it is the gravity, (see Figure D.1 at time t_0). Without any force added to the system, we will observe that over time the gum will start to deform, (see Figure D.1 during time $t_1 \to t_3$). After it has been deforming for a while, we cut its end off, *i.e.* we remove the force, and we will see that part of the deformation recovers instantly, and we will also verify that over time another part of the deformation recovers slowly.

That is, these materials have the ability to store mechanical energy as elastic solids and can also dissipate energy due to their viscosity. Hence, when we are working with how to approach the constitutive equation for these materials we have to take into account these phenomena simultaneously, (see Findley *et al.* (1976), Christensen (1982)).

In other words, viscoelastic materials are those in which the stress-strain relationship is time dependent. The most relevant viscoelastic phenomena are listed below:

Creep – When stress is constant, strain increases over time. For example we can mention a building column, which, when force is first applied shows an initial strain, which increases over time with no corresponding increase in stress, (see Figure D.2).

Relaxation – When strain is constant, stress decreases over time. As an example we can cite a prestressed cable bridge whose cable is initially subjected to an initial strain causing an initial stress and over time this stress decreases while the strain remains constant, (see Figure D.3).

On a final note, creep and relaxation are reciprocal phenomena.

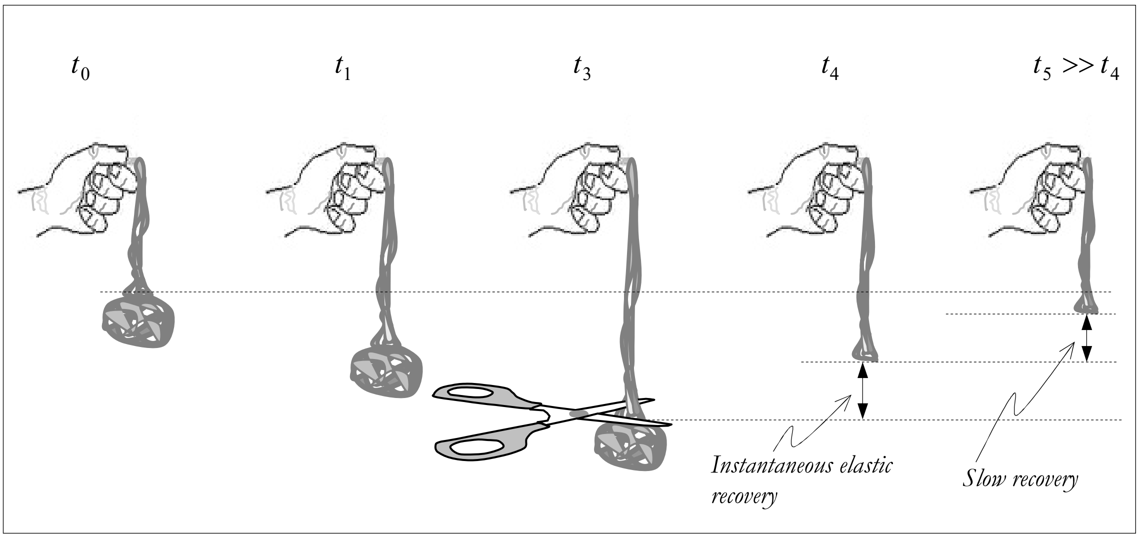

Figure D.1: Viscoelastic behavior.

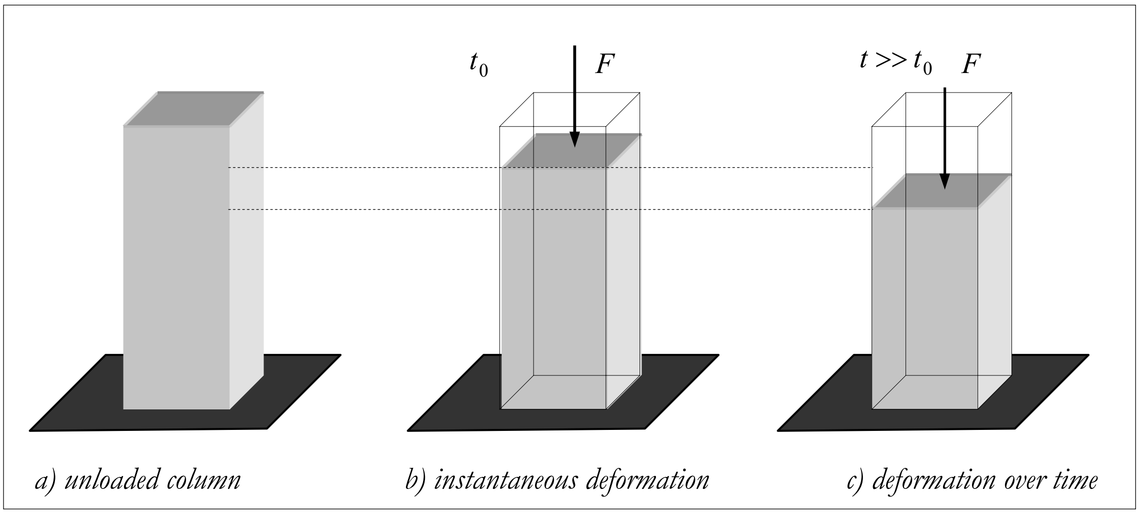

Figure D.2: Creep phenomenon.

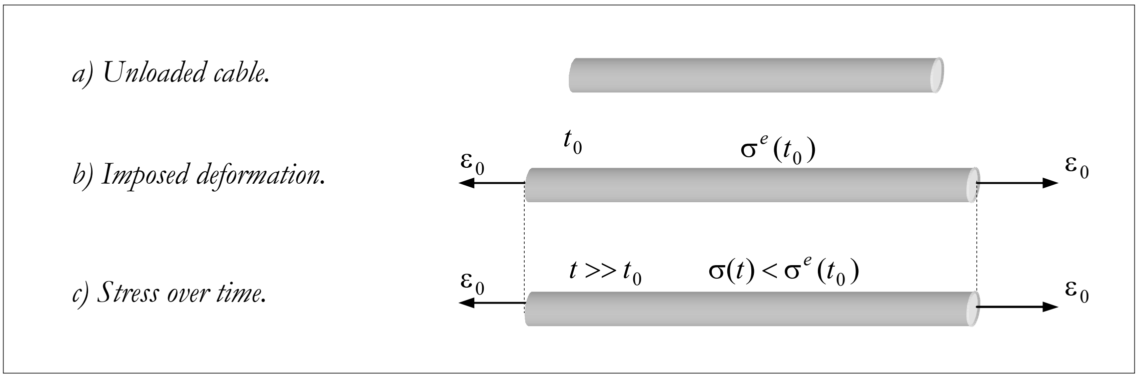

Figure D.3: Relaxation phenomenon.

As known, elastic materials have the capacity to store mechanical energy. In contrast, Newtonian viscous fluids in motion are associated with energy dissipation due to viscosity, in other words, fluids have no capacity to store energy. This Annex will deal with a material that exhibits behavior of elastic solid and fluid simultaneously. In other words, this material has the capacity to store mechanical energy according to solid elastic law and has the dissipation energy according the fluid law. This material is called *Viscoelastic Materials*, (Findley *et al.* (1976), Christensen (1982)).

Viscoelastic materials are characterized by presenting dissipative process (irreversible). In order to predict the stress/strain over time it is necessary to adopt a constitutive law which has this phenomenon in consideration. Next, we will expose some stress/strain responses related to viscoelastic material behavior, e.g.:

- Instantaneous linear elasticity;
- Creep under constant stress;
- Relaxation of stress under constant strain;
- Slow recovery;
- Permanent state.

- *Instantaneous linear elastic behavior*

By applying a initial stress σ_0 an instantaneous strain ε_0 appears which disappears when at time t_1 when the stress is removed, (see Figure D.4). This behavior is governed by Hook's law (linear elasticity), and the strain for one-dimensional problem is given by $\varepsilon_0 = \frac{\sigma_0}{E}$.

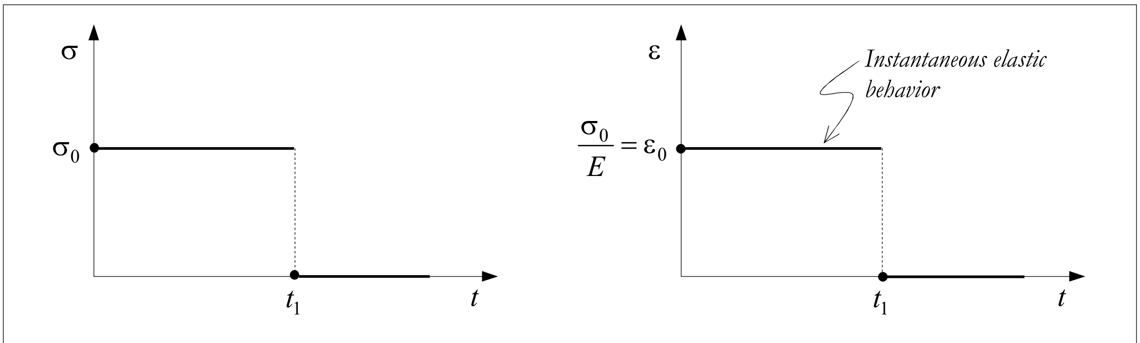

Figure D.4: Instantaneous elastic behavior.

- *Creep under constant stress*

Now consider that at time t_1 it is applied a stress σ_0 which is maintained constant over time. At the initial time t_1 an instantaneous strain ε_0 appears and as time goes on it can be observed a variation in strain although the stress is maintained constant, (see Figure D.5).

The applied stress $\sigma(t)$ can be represented by means of the Heaviside step function defined as follows:

$$H(t-\tau) = \begin{cases} 0 & \forall t < t_1 \\ 1 & \forall t \geq t_1 \end{cases} \quad \textit{Heaviside step function} \tag{D.1}$$

Then the stress state can be represented by:

$$\sigma = \sigma_0 H(t - t_1) \tag{D.2}$$

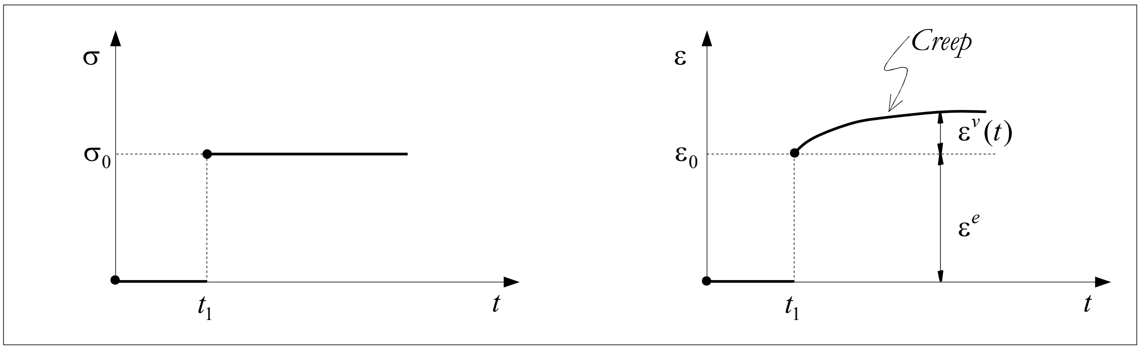

Figure D.5: Creep under constant stress.

In this scenario, the strain state can be split additively into an elastic and a viscoelastic part:

$$\varepsilon = \varepsilon^e + \varepsilon^v \tag{D.3}$$

- *Stress Relaxation under constant strain*

Now consider that an initial strain ε_0 is applied at time t_1, and ε_0 is maintained constant over time. At time t_1 an initial stress σ_0 appears, and as time goes on it can be observed a variation in stress, (see Figure D.6). For this scenario, the strain state can be represented by means of Heaviside step function:

$$\varepsilon = \varepsilon_0 H(t - \tau) \tag{D.4}$$

And the stress state can be split additively into an elastic and a viscoelastic part:

$$\sigma(t) = \sigma^e + \sigma^v(t) \tag{D.5}$$

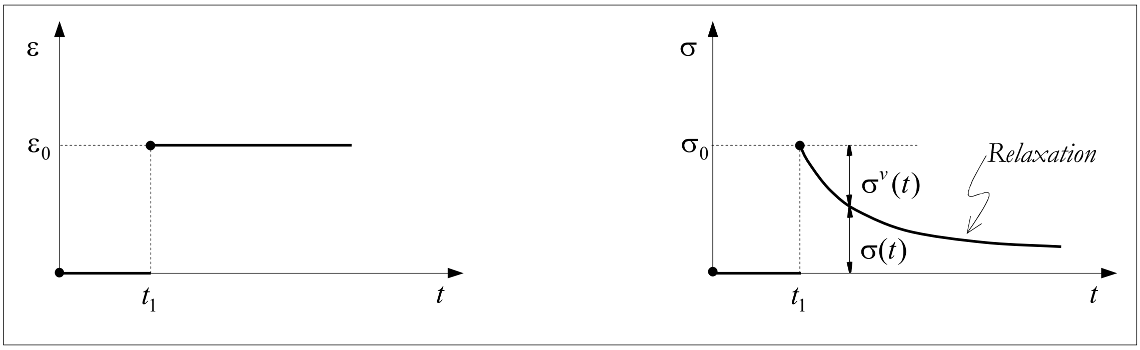

Figure D.6: Stress relaxation under constant strain.

- *Slow recovery and Permanent state*

Let us suppose now that the material undergoes the following behavior: at time t_1 an initial stress σ_0 is applied until time t_2, during this time interval it can be observed a variation in strain due creep phenomenon, (see Figure D.7). At time t_2 the stress is removed and it can be observed that there is an instantaneous strain recovery and during the time interval t_2 - t_3 there is a slow strain recovery after that a residual strain is observed.

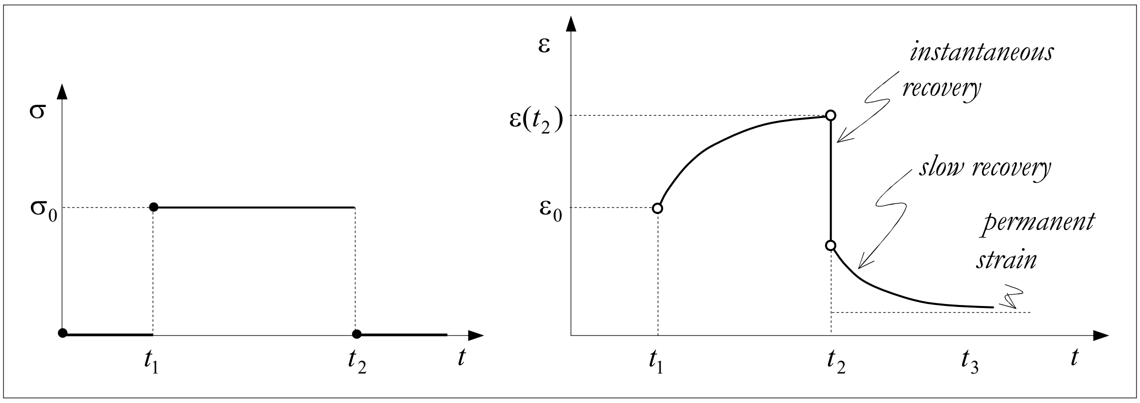

Figure D.7: Instantaneous and slow recovery.

D.2 Viscoelasticity Rheological Models

In order to formulate the phenomena described previously we will use some mechanical devices called rheological models. The rheological models are mechanical devices which are useful in order to interpret and to establish the mathematical equations for certain physical phenomenon. In viscoelasticity those models are made up by simple devices such as: spring and dashpot. Several viscoelastic models can be established by the combinations of these devices.

The spring device represents the linear elastic model, and the spring constant is represented by E (Young's modulus), (see Figure D.8(a)). The stress in the spring is represented as follows:

$$\sigma = E\varepsilon \tag{D.6}$$

The spring device represents the instantaneous linear elastic behavior, (see Figure D.4).

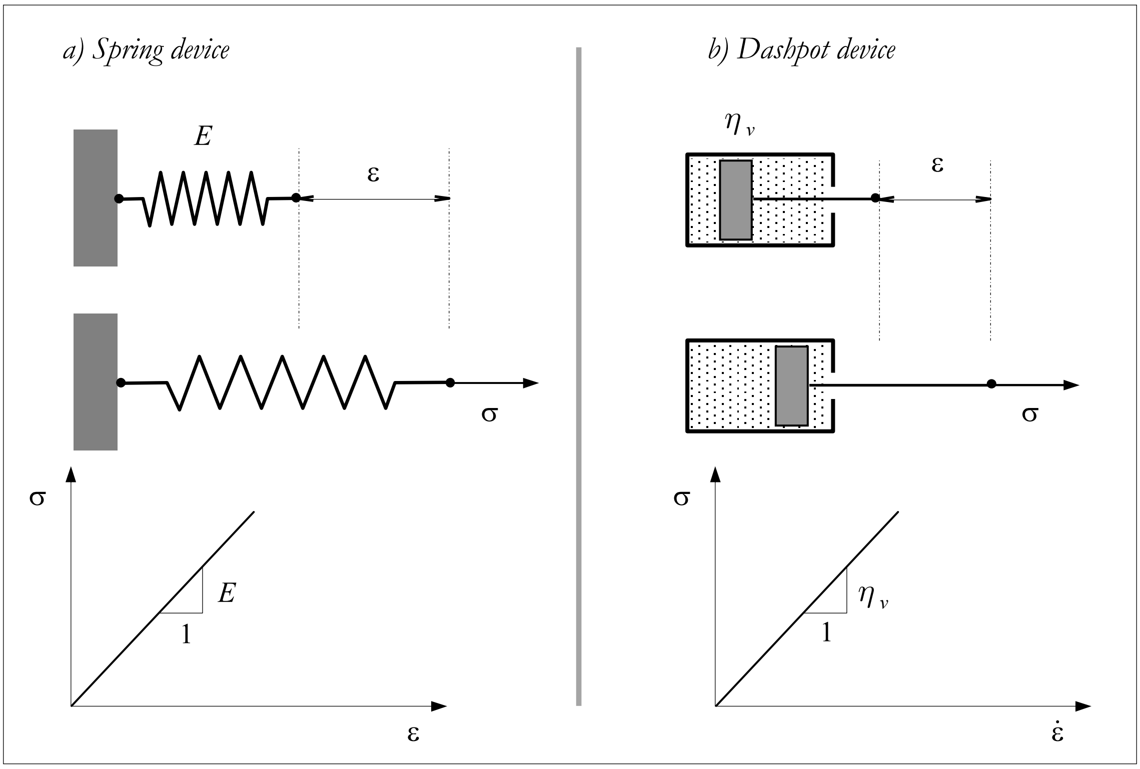

Figure D.8: Rheological models.

In order to represent the viscous model a dashpot is employed with viscoelastic parameter represented by η_v (viscoelastic coefficient), (see Figure D.8(b)). The stress in the dashpot is represented by:

$$\sigma = \eta_v \dot{\varepsilon} \tag{D.7}$$

where $\dot{\varepsilon} \equiv \frac{D\varepsilon}{Dt}$, is the rate of change of strain. The equation (D.7) establishes that $\dot{\varepsilon}$ is linearly proportional to the stress, in other words, the dashpot will deform continuously in a constant rate when the material is subjected to a constant stress, (see Figure D.9).

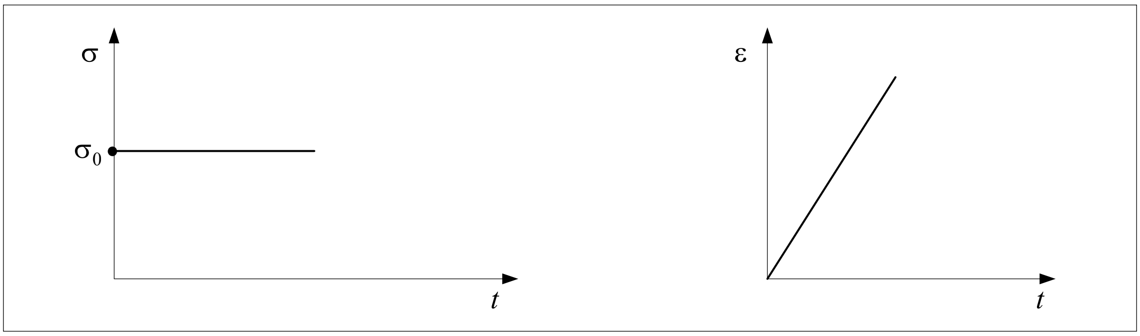

Figure D.9: Dashpot device behavior.

On other hand, when an instantaneous strain is applied in the dashpot device it will appear an infinite stress in the device and the stress will decrease rapidly until reach the value $\sigma = 0$, (see Figure D.10).

Mathematically the stress can be represented by Dirac delta function $\delta(t)$; $\delta(t)=0$ when $t\neq 0$ and $\delta(t)=\infty$ when $t=0$. Then, the stress due to an impose strain ε_0 can be represented as follows:

$$\sigma(t)=\eta_v\varepsilon_0\delta(t) \tag{D.8}$$

Physically the infinite stress is unrealistic, since is also unrealistic to apply an instantaneous strain in the dashpot.

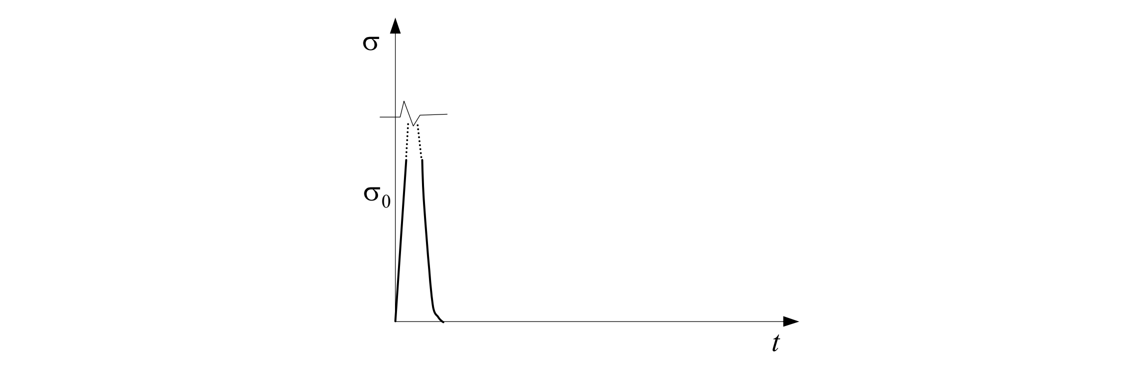

Figure D.10: Stress behavior due to an instantaneous strain in the dashpot device.

D.3 Viscoelastic Models

Next, we will introduce two one-dimensional viscoelastic models, namely: Maxwell model and the Kelvin model (also called Voigt model). These models separately do not represent accurately the viscoelastic material behavior; nevertheless the combinations of these models could be acceptable in order to characterize the viscoelastic behavior.

The Maxwell Model

The Maxwell model is made up of a spring and a dashpot in series as indicated in Figure D.11.

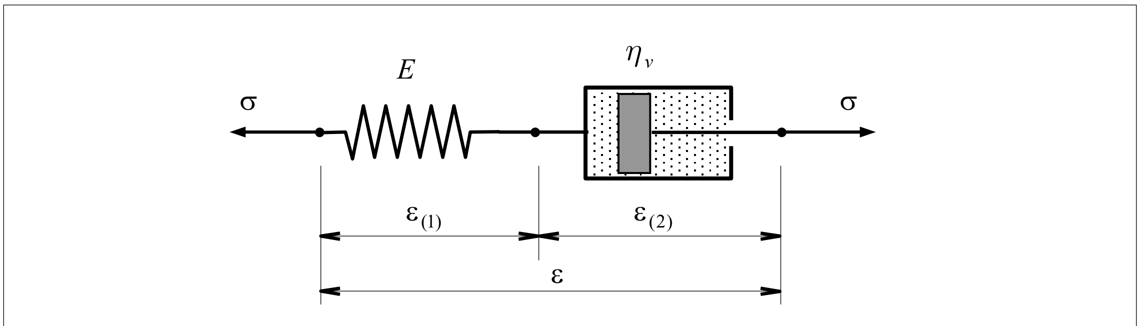

Figure D.11: The Maxwell model.

The stress in the spring is given by:

$$\sigma=E\varepsilon_{(1)} \xrightarrow{\text{Rate of change}} \dot{\sigma}=E\dot{\varepsilon}_{(1)} \quad\Rightarrow\quad \dot{\varepsilon}_{(1)}=\frac{\dot{\sigma}}{E} \tag{D.9}$$

and in the dashpot is given by:

$$\sigma = \eta_v \dot{\varepsilon}_{(2)} \quad \Rightarrow \quad \dot{\varepsilon}_{(2)} = \frac{\sigma}{\eta_v} \tag{D.10}$$

Since both devices are in series the total strain is obtained by the sum of strain of each element, i.e.:

$$\varepsilon = \varepsilon_{(1)} + \varepsilon_{(2)} \tag{D.11}$$

and its rate of change is given by:

$$\dot{\varepsilon} = \dot{\varepsilon}_{(1)} + \dot{\varepsilon}_{(2)} \tag{D.12}$$

The stress-strain relationship can be obtained by substituting $\dot{\varepsilon}_{(1)} = \frac{\dot{\sigma}}{E}$ and $\dot{\varepsilon}_{(2)} = \frac{\sigma}{\eta_v}$ into the equation (D.12), with that we can obtain:

$$\boxed{\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta_v}} \quad \textit{Constitutive Law for Maxwell Model} \tag{D.13}$$

By means of the linear differential operator $\frac{d}{dt}$ the constitutive law for Maxwell model can also be represented as follows:

$$\frac{d}{dt}\varepsilon = \left(\frac{1}{E}\frac{d}{dt} + \frac{1}{\eta_v} \right)\sigma \tag{D.14}$$

The stress-strain relations for several stress state can be obtained by means of the solution of the differential equation (D.13). For example, by applying the constant stress $\sigma = \sigma_0$ at time $t = 0$, the equation (D.13) becomes a first order differential equation which solution is:

$$\varepsilon(t) = \frac{\sigma_0}{E} + \frac{\sigma_0}{\eta_v} t \tag{D.15}$$

If the stress is removed at time t_1 the whole deformation in the spring is recovered $\varepsilon_{(1)} = \frac{\sigma_0}{E}$ and a residual strain $\frac{\sigma_0}{\eta_v} t_1$ is kept in the dashpot, (see Figure D.12).

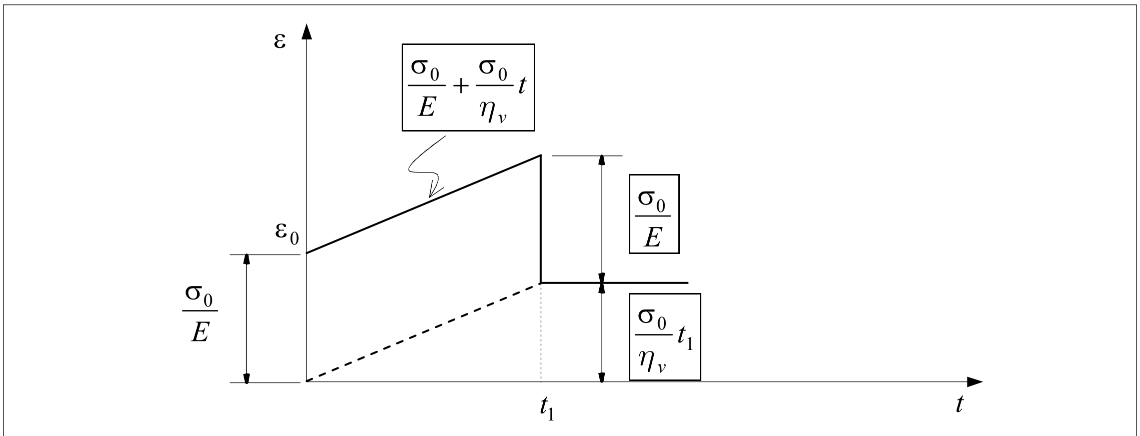

Figure D.12: Dashpot behavior.

If the Maxwell model is subjected to a constant strain ε_0 at time t_0, then the instantaneous stress σ_0 appears in the spring device. As time goes on the Maxwell model undergoes a stress relaxation due to the dashpot. The stress can be obtained by integrating the equation (D.13), i.e.:

$$\sigma(t) = \sigma_0 \exp^{\left(\frac{-Et}{\eta_v}\right)} = E\varepsilon_0 \exp^{\left(\frac{-Et}{\eta_v}\right)} \tag{D.16}$$

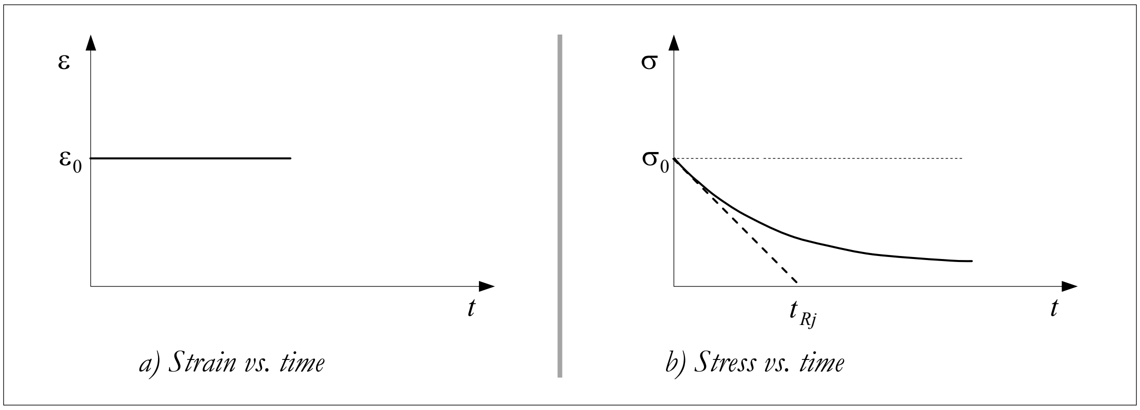

Figure D.13: Relaxation time.

The equation (D.16) describes the relaxation phenomenon in the Maxwell model under constant strain. The rate of change of the equation (D.16) is given by:

$$\dot{\sigma} = \sigma_0 \left(\frac{-E}{\eta_v}\right) \exp^{\left(\frac{-Et}{\eta_v}\right)} = -\left(\frac{\sigma_0 E}{\eta_v}\right) \exp^{\left(\frac{-Et}{\eta_v}\right)} \tag{D.17}$$

And by integrating we can obtain:

$$\sigma = -\frac{\sigma_0 E}{\eta_v} t + \sigma_0 \tag{D.18}$$

According to the above equation, the stress will be zero when the equation $t = \frac{\eta_v}{E} \equiv t_{Rj}$ holds, with that we define t_{Rj} as the *relaxation time.*

The relaxation time is a mechanical property of viscoelastic materials. In fact, most of the stress relaxation occurs during the time t_{Rj}, (see Figure D.13). Then, each material presents its own relaxation time. A short relaxation time implies that the stress decreases very fast, in the contrast for long relaxation the stress decreases slowly, (see Figure D.14).

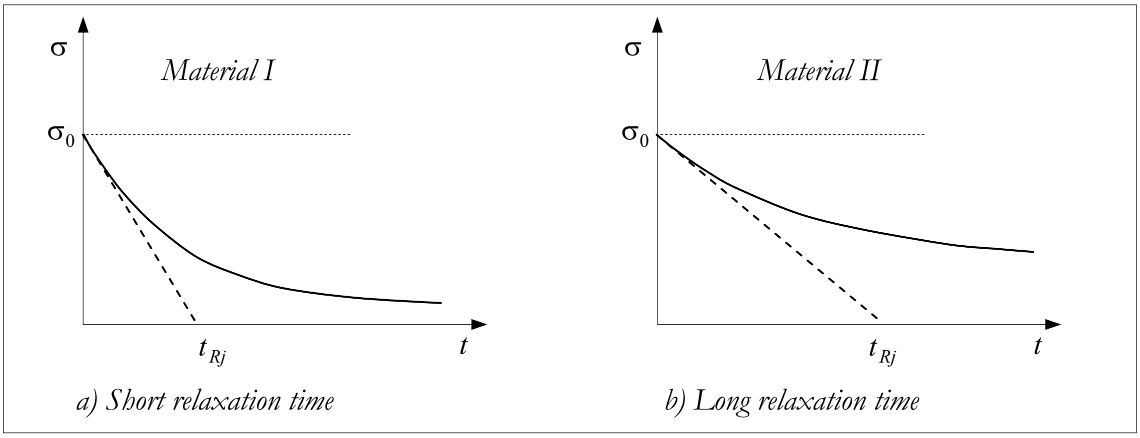

Figure D.14: Relaxation time for different materials.

Kelvin Model (Voigt Model)

The Kelvin model, also called Voigt model, is made up of a spring and a dashpot arranged in parallel as indicated in Figure D.15. This model establishes that the strain either in spring or in the dashpot is the same, i.e.:

$$\varepsilon_{(1)} = \varepsilon_{(2)} = \varepsilon \tag{D.19}$$

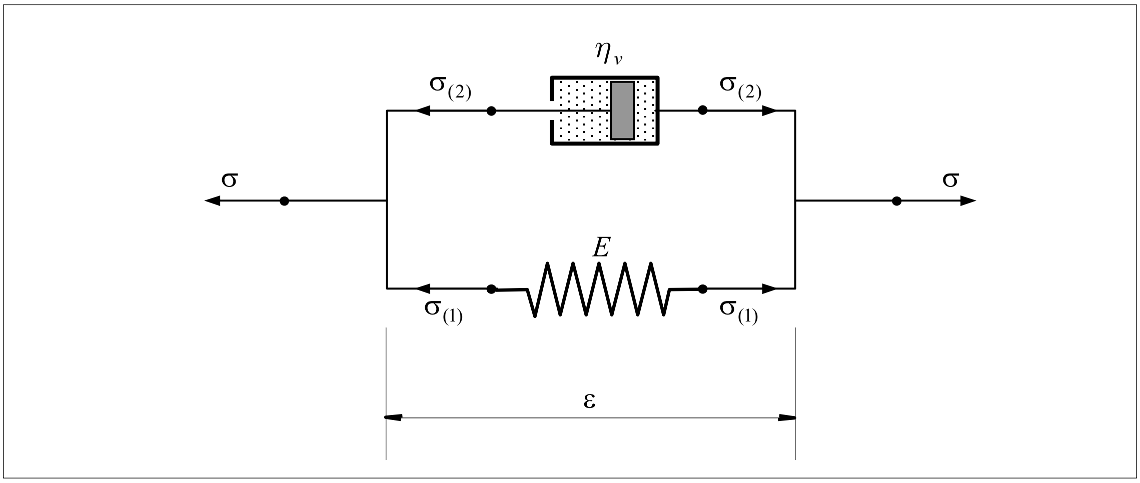

Figure D.15: Kelvin model.

The stresses in the spring and in the dashpot are given respectively by:

$$\sigma_{(1)} = E\varepsilon \quad ; \quad \sigma_{(2)} = \eta_v \dot{\varepsilon} \tag{D.20}$$

Since the devices are in parallel the following is true:

$$\sigma = \sigma_{(1)} + \sigma_{(2)} \tag{D.21}$$

By substituting the equations in (D.20) into the equation (D.21) we can obtain the stress-strain relationship for the Kelvin model:

$$\boxed{\sigma = E\varepsilon + \eta_v \dot{\varepsilon}} \quad \textit{Constitutive law for Kelvin model} \tag{D.22}$$

By means of the linear differential operator $\frac{d}{dt}$ the constitutive law for the Kelvin model can also be written as follows:

$$\sigma = \left(E + \eta_v \frac{d}{dt} \right) \varepsilon \tag{D.23}$$

By solving the differential equation (D.22) we can obtain:

$$\varepsilon(t) = \frac{\sigma_0}{E} \left[1 - \exp^{\left(\frac{-Et}{\eta_v} \right)} \right] \tag{D.24}$$

Note that for a constant stress the strain will increase exponentially over time. The graphical representation for the function (D.24) can be appreciated in Figure D.16.

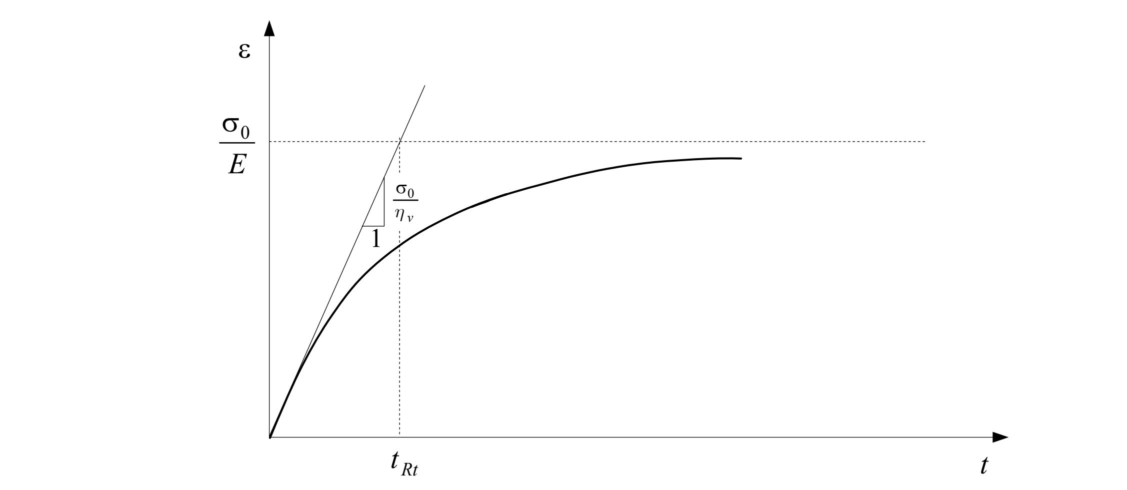

Figure D.16: Strain function for the Kelvin model. Delayed time t_{Rt}.

The rate of change of the equation (D.24) provides the slope of the curve $\varepsilon(t)$:

$$\dot{\varepsilon} = \frac{\sigma_0}{\eta_v} \exp^{\left(\frac{-Et}{\eta_v}\right)} \tag{D.25}$$

Note that when $t \to \infty$, $\dot{\varepsilon} = 0$ and at $t = 0$ the initial slope is given by $\dot{\varepsilon} = \frac{\sigma_0}{\eta_v}$ and the time related to this slope when intercept the line $\varepsilon = \frac{\sigma_0}{E}$ is called *delayed time* $t_{Rt} = \frac{\eta_v}{E}$. In fact, most of the total strain $\frac{\sigma_0}{\eta_v}$ occurs at the delayed time, since $\exp^{\left(\frac{-Et}{\eta_v}\right)}$ converges very fast when $t < t_{Rt}$. At time $t = t_{Rt}$ the strain is given by $\varepsilon = \frac{\sigma_0}{E}\left(1 - \frac{1}{\mathrm{e}}\right) = 0.63\frac{\sigma_0}{E}$, i.e. 63% of the total strain.

Now, if the stress is removed at time t_1 the strain can be obtained by means of the superposition principle. The strain $\varepsilon_{(a)}$ due to the stress σ_0 (applied at time $t = 0$) is given by:

$$\varepsilon_{(a)} = \frac{\sigma_0}{E}\left[1 - \exp^{\left(\frac{-Et}{\eta_v}\right)}\right] \tag{D.26}$$

The strain $\varepsilon_{(b)}$ due to the stress $(-\sigma_0)$ as time t_1 is:

$$\varepsilon_{(b)} = -\frac{\sigma_0}{E}\left[1 - \exp^{\left(\frac{-E(t-t_1)}{\eta_v}\right)}\right] \tag{D.27}$$

Then, by applying the superposition principle we can obtain:

$$\varepsilon = \varepsilon_{(a)} + \varepsilon_{(b)} = \frac{\sigma_0}{E}\exp^{\left(\frac{-Et}{\eta_v}\right)}\left[\exp^{\left(\frac{-Et_1}{\eta_v}\right)} - 1\right] \qquad t > t_1 \tag{D.28}$$

The graphical representation for both states can be appreciated in Figure D.17.

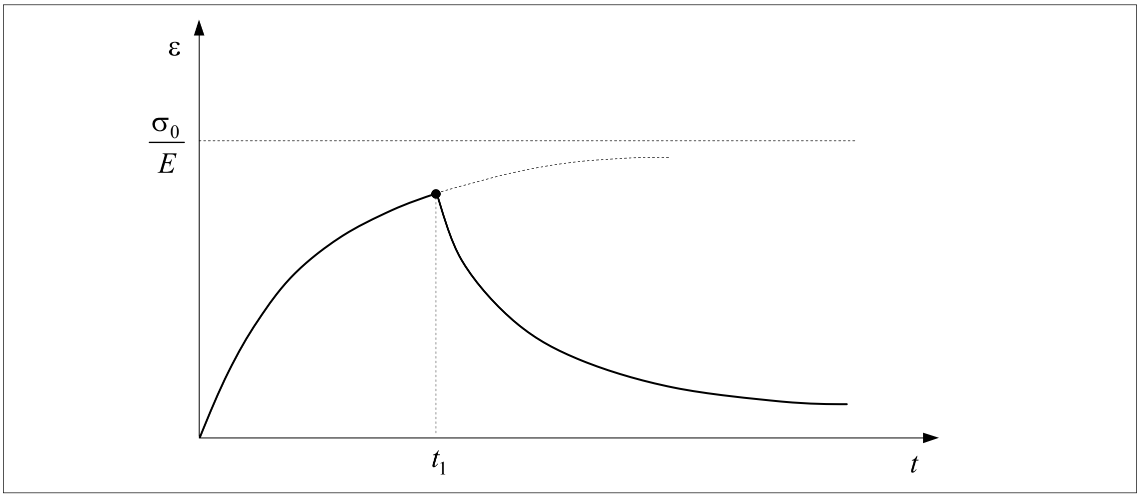

Figure D.17: Modelo de Kelvin – Loading unloading processes.

Burgers Model

The Burgers model (also called four devices model) is made up of the Maxwell model and the Kelvin model arranged in series, (see Figure D.18). The Burgers model is able to characterize three viscoelastic behaviors, namely: an *instantaneous elastic response* due to the spring of parameter E_1; a *viscous flow* due to the dashpot of parameter η_{v1}; and a *delayed elastic* response due to the Kelvin model.

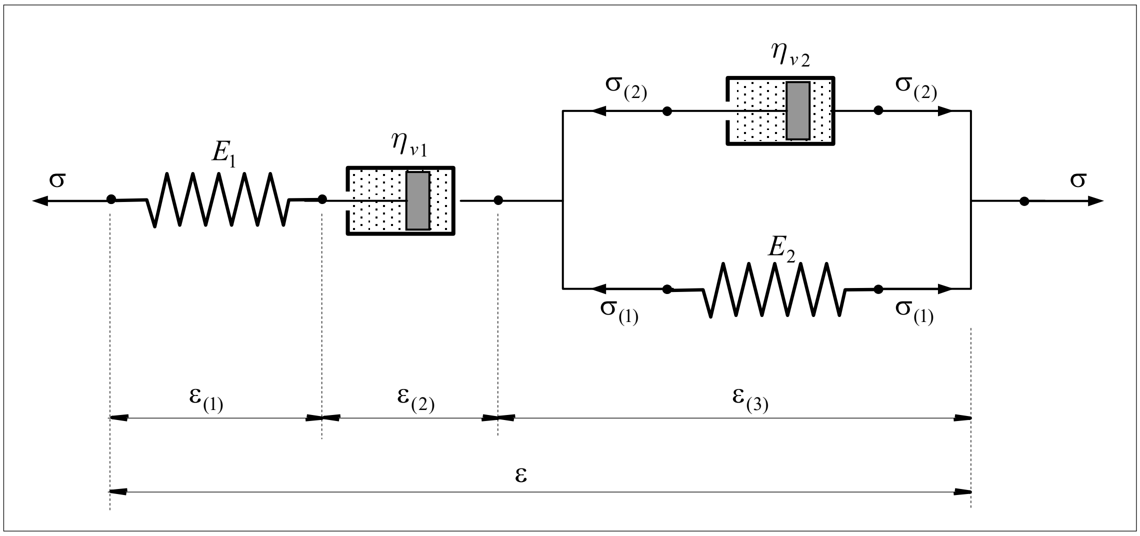

Figure D.18: Burgers model.

The constitutive equation for the Burgers model can be obtained by taking into account the strain, under a constant stress, for each device coupled in series, (see Figure D.18). Then, the total strain at time t can be obtained as follows:

$$\varepsilon = \varepsilon_{(1)} + \varepsilon_{(2)} + \varepsilon_{(3)} \tag{D.29}$$

where

$$\varepsilon_{(1)} = \frac{\sigma}{E_1} \quad ; \quad \dot{\varepsilon}_{(2)} = \frac{\sigma}{\eta_{v1}} \tag{D.30}$$

and $\varepsilon_{(3)}$ is the strain that undergoes the Kelvin model, (see equation (D.22)):

$$\dot{\varepsilon}_{(3)} + \frac{E_2}{\eta_{v2}}\varepsilon_{(3)} = \frac{\sigma}{\eta_{v2}} \quad \Rightarrow \quad \dot{\varepsilon}_{(3)} = \frac{\sigma}{\eta_{v2}} - \frac{E_2}{\eta_{v2}}\varepsilon_{(3)} \tag{D.31}$$

In order to obtain the constitutive equation for this model we take the rate of change of the equation in (D.29) and we substitute the respective rates of changes of $\varepsilon_{(1)}$, $\varepsilon_{(2)}$ and $\varepsilon_{(3)}$, i.e.:

$$\dot{\varepsilon} = \dot{\varepsilon}_{(1)} + \dot{\varepsilon}_{(2)} + \dot{\varepsilon}_{(3)} = \frac{\dot{\sigma}}{E_1} + \frac{\sigma}{\eta_{v1}} + \left(\frac{\sigma}{\eta_{v2}} - \frac{E_2}{\eta_{v2}}\varepsilon_{(3)}\right) \tag{D.32}$$

We take the rate of change of the above equation in order to obtain:

$$\ddot{\varepsilon} = \frac{\ddot{\sigma}}{E_1} + \frac{\dot{\sigma}}{\eta_{v1}} + \left(\frac{\dot{\sigma}}{\eta_{v2}} - \frac{E_2}{\eta_{v2}}\dot{\varepsilon}_{(3)}\right) \tag{D.33}$$

Taking into account $\dot{\varepsilon}_{(3)} = \dot{\varepsilon} - \dot{\varepsilon}_{(1)} - \dot{\varepsilon}_{(2)} = \dot{\varepsilon} - \frac{\dot{\sigma}}{E_1} - \frac{\sigma}{\eta_{v1}}$ into the above equation we can obtain:

$$\begin{aligned}
\ddot{\varepsilon} &= \frac{\ddot{\sigma}}{E_1} + \frac{\dot{\sigma}}{\eta_{v1}} + \frac{\dot{\sigma}}{\eta_{v2}} - \frac{E_2}{\eta_{v2}}\left(\dot{\varepsilon} - \frac{\dot{\sigma}}{E_1} - \frac{\sigma}{\eta_{v1}}\right) \\
&= \frac{\ddot{\sigma}}{E_1} + \left(\frac{1}{\eta_{v1}} + \frac{1}{\eta_{v2}}\right)\dot{\sigma} - \frac{E_2}{\eta_{v2}}\dot{\varepsilon} + \frac{E_2}{\eta_{v2}}\frac{\dot{\sigma}}{E_1} + \frac{E_2}{\eta_{v2}}\frac{\sigma}{\eta_{v1}} \\
&= \frac{\ddot{\sigma}}{E_1} + \left(\frac{1}{\eta_{v1}} + \frac{1}{\eta_{v2}} + \frac{E_2}{\eta_{v2}E_1}\right)\dot{\sigma} - \frac{E_2}{\eta_{v2}}\dot{\varepsilon} + \frac{E_2}{\eta_{v2}}\frac{\sigma}{\eta_{v1}}
\end{aligned} \tag{D.34}$$

By multiplying each term of the above equation by $\frac{\eta_{v2}\eta_{v1}}{E_2}$ we can obtain the constitutive equation for the Burgers model:

$$\sigma + \left(\frac{\eta_{v1}}{E_1} + \frac{\eta_{v1}}{E_2} + \frac{\eta_{v2}}{E_2}\right)\dot{\sigma} + \frac{\eta_{v1}\eta_{v2}}{E_1 E_2}\ddot{\sigma} = \eta_{v1}\dot{\varepsilon} + \frac{\eta_{v1}\eta_{v2}}{E_2}\ddot{\varepsilon} \tag{D.35}$$

Constitutive equations for the Burgers model

The stress-strain relationship for the model with three or four parameters can be expressed generically as follows:

$$p_0\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon} + q_2\ddot{\varepsilon} \tag{D.36}$$

where the parameters p_0, p_1, p_2, q_0, q_1 and q_2 can be obtained according to the number and the spring arrangements.

The creep phenomenon, under constant stress, for the Burgers model can be obtained by means of the solution of the differential equation (D.35) with the following initial conditions:

$$\begin{aligned} &\varepsilon = \varepsilon_{(1)} = \frac{\sigma_0}{E_1} \quad ; \quad \varepsilon_{(2)} = \varepsilon_{(3)} = 0 \quad at \quad t=0 \\ &\dot{\varepsilon} = \frac{\sigma_0}{\eta_{v1}} + \frac{\sigma_0}{\eta_{v2}} \quad at \quad t=0 \end{aligned} \tag{D.37}$$

By means of Laplace transformation the differential equation solution can be obtained:

$$\varepsilon(t) = \frac{\sigma_0}{E_1} + \frac{\sigma_0}{\eta_{v1}} t + \frac{\sigma_0}{E_2}\left(1 - \exp^{\frac{-E_2 t}{\eta_{v2}}}\right) \tag{D.38}$$

By comparing the above equation (D.38) with the equations (D.15) and (D.24) we can conclude that the creep phenomenon, for the Burgers model, is the sum of the Maxwell model creep phenomenon and the Kelvin model creep phenomenon, where we can identify the following terms: $\frac{\sigma_0}{E_1}$ - instantaneous elastic strain; $\frac{\sigma_0}{\eta_{v1}} t$ - viscous flow; and $\frac{\sigma_0}{E_2}\left(1 - \exp^{\frac{-E_e t}{\eta_{v2}}}\right)$ - delayed elastic process.

By taking the rate of change of (D.38) we can obtain the slope of the curve $\varepsilon \times t$:

$$\dot{\varepsilon}(t) = \frac{\sigma_0}{\eta_{v1}} + \frac{\sigma_0}{\eta_{v2}} \exp^{\frac{-E_2 t}{\eta_{v2}}} \tag{D.39}$$

Then, the slope of the curve $\varepsilon \times t$ at time $t = 0^+$, (see Figure D.19), is given by:

$$\dot{\varepsilon}(t = 0^+) = \left(\frac{1}{\eta_{v1}} + \frac{1}{\eta_{v2}}\right)\sigma_0 = \tan(\alpha) \tag{D.40}$$

The curve $\varepsilon \times t$ has an asymptotic line, (see Figure D.19), with slope given by:

$$\dot{\varepsilon}(t \to \infty) = \frac{\sigma_0}{\eta_{v1}} = \tan(\beta) \tag{D.41}$$

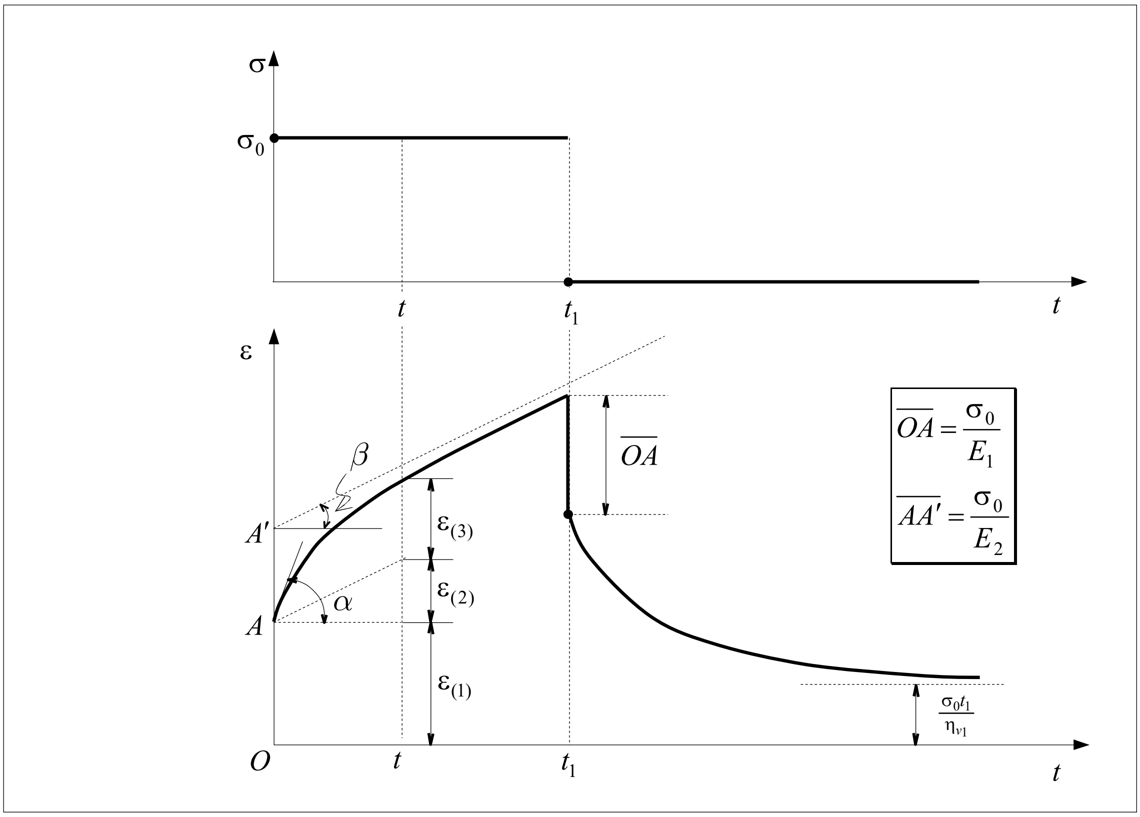

Figure D.19: Creep phenomenon for the Burgers model.

D.4 Generalization of the Maxwell and Kelvin Models

It can be appreciated in Figure D.20 the generalization of the Maxwell model in series.

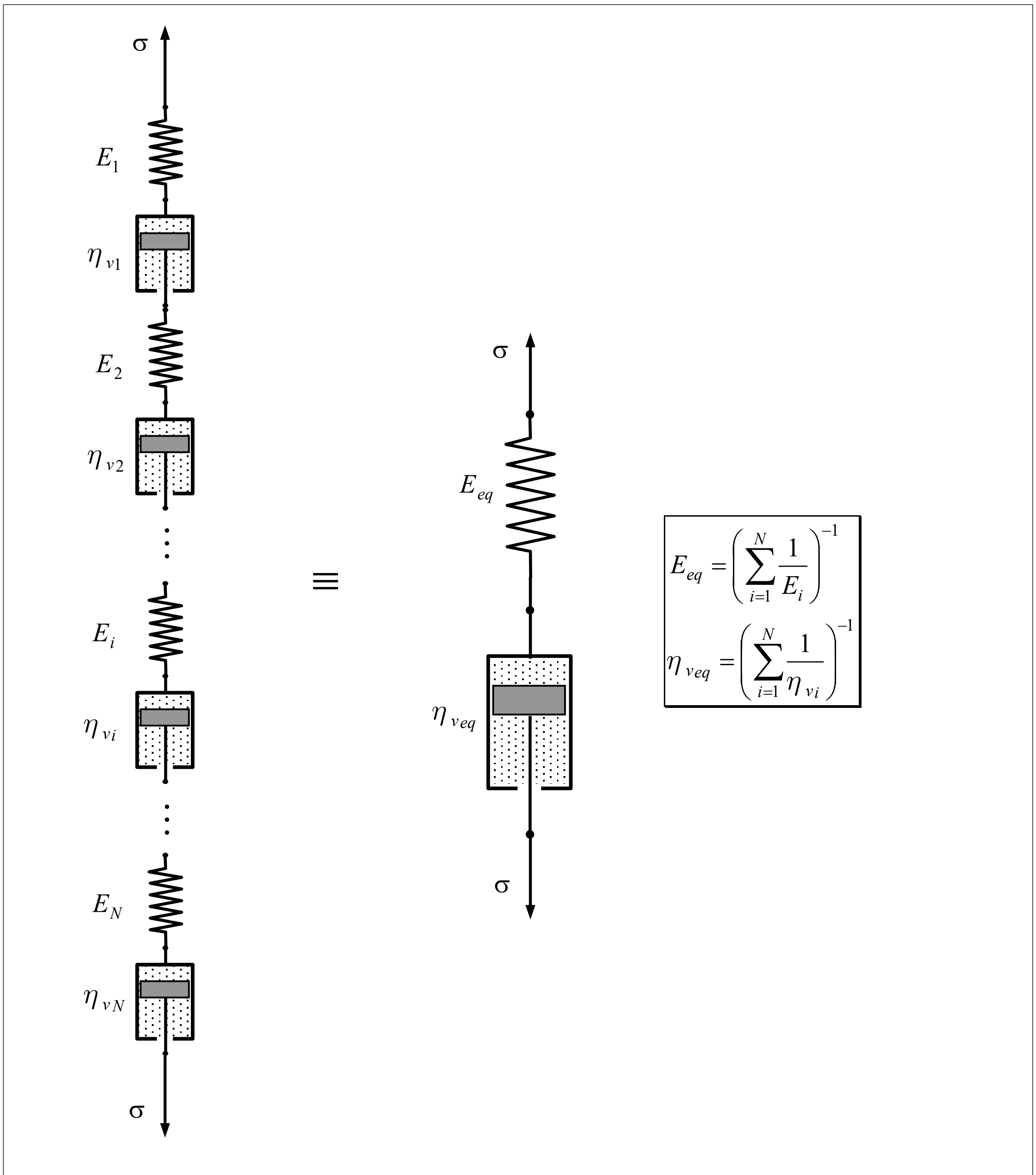

Figure D.20: Generalization of the Maxwell model in series.

The constitutive equation for the model presented in Figure D.20 is given by:

$$\dot{\varepsilon} = \dot{\sigma}\sum_{i=1}^{N}\frac{1}{E_i} + \sigma\sum_{i=1}^{N}\frac{1}{\eta_{vi}} \tag{D.42}$$

The above equation is equivalent to the equation in (D.13) and describes the same mechanics.

If the Kelvin models are arranged in parallel, (see Figure D.10), the constitutive equation for this generalized model is given by:

$$\sigma = \varepsilon\sum_{i=1}^{N} E_i + \dot{\varepsilon}\sum_{i=1}^{N}\eta_{vi} \tag{D.43}$$

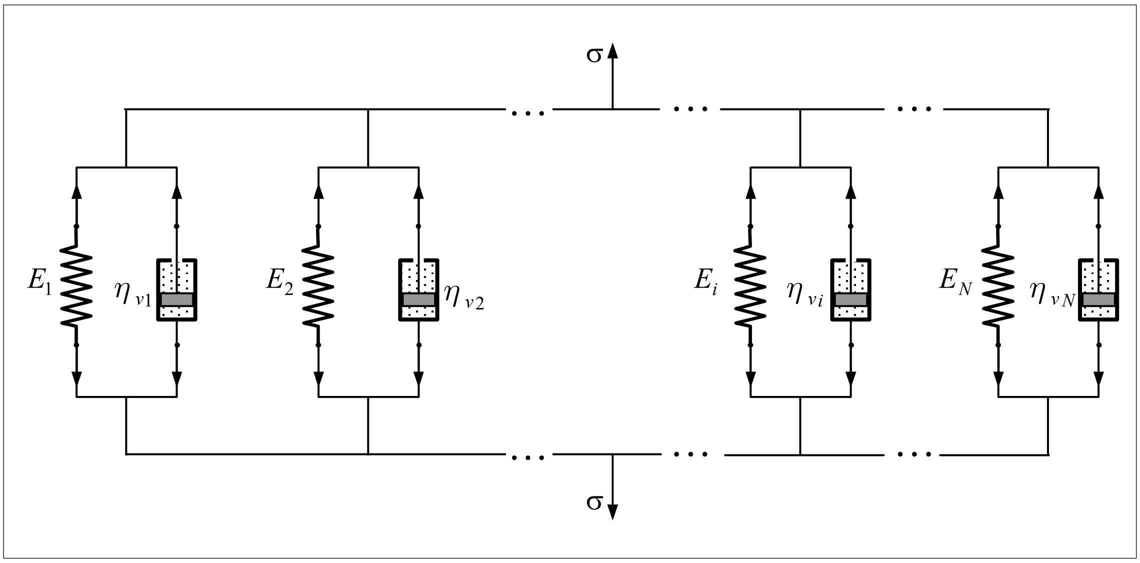

Figure D.21: Generalization of the Kelvin models arranged in parallel.

When several Maxwell models are arranged in parallel, this model can characterize: the instantaneous elastic process; delayed process with several delayed times; stress relaxation with several relaxation times; and also viscous flow, (see Figure D.22).

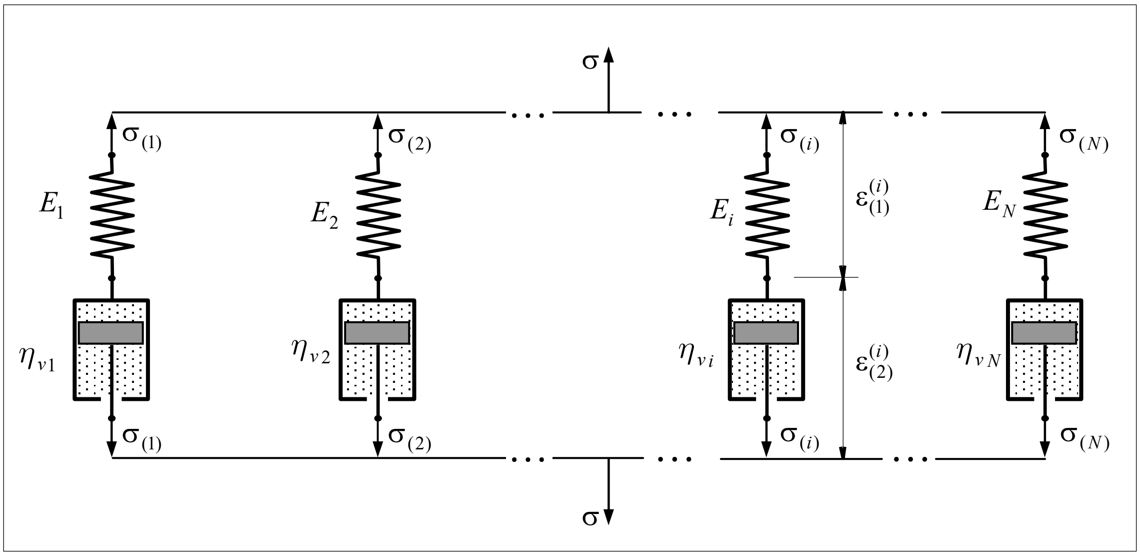

Figure D.22: Generalization of the Maxwell models arranged in parallel.

In order to predict the stress associated with variation of prescribed strain, the generalization of the Maxwell model is not convenient since the same prescribed strain is applied for each device individually, and also the stress is the sum of the contribution of each element.

For a generic element i, (see Figure D.22), it fulfills that:

$$\varepsilon = \varepsilon_{(1)}^{(i)} + \varepsilon_{(1)}^{(i)} \qquad \Rightarrow \qquad \dot{\varepsilon} = \dot{\varepsilon}_{(1)}^{(i)} + \dot{\varepsilon}_{(2)}^{(i)} \tag{D.44}$$

In the same fashion as the equation in (D.13) we can establish that:

$$\dot{\varepsilon} = \dot{\varepsilon}_{(1)}^{(i)} + \dot{\varepsilon}_{(2)}^{(i)} = \frac{\dot{\sigma}_{(i)}}{E_i} + \frac{\sigma_{(i)}}{\eta_{vi}} = \left[\frac{1}{E_i}\frac{d}{dt} + \frac{1}{\eta_{vi}}\right]\sigma_{(i)} \tag{D.45}$$

or

$$\sigma_{(i)} = \left[\frac{1}{\frac{1}{E_i}\frac{d}{dt} + \frac{1}{\eta_{vi}}}\right]\dot{\varepsilon} \tag{D.46}$$

Then, the total stress, due to the contribution of each element, is given by:

$$\sigma = \sum_{i=1}^{N}\sigma_i = \left(\sum_{i=1}^{N}\frac{1}{\frac{1}{\eta_{vi}} + \frac{1}{E_i}\frac{d}{dt}}\right)\dot{\varepsilon} \tag{D.47}$$

By multiplying by $\prod_{i=1}^{a}\left(\frac{1}{E_i}\frac{d}{dt} + \frac{1}{\eta_{vi}}\right)$ both sides of the above equation and also by discarding the operator $\prod_{i=1}^{a}$ we can obtain:

$$\left[\left(\frac{1}{E_1}\frac{d}{dt} + \frac{1}{\eta_{v1}}\right)\left(\frac{1}{E_2}\frac{d}{dt} + \frac{1}{\eta_{v2}}\right)\cdots\right]\sigma =$$
$$\left[\left(\frac{1}{E_1}\frac{d}{dt} + \frac{1}{\eta_{v1}}\right)\left(\frac{1}{E_2}\frac{d}{dt} + \frac{1}{\eta_{v2}}\right)\cdots\right]\left[\frac{1}{\frac{1}{E_1}\frac{d}{dt} + \frac{1}{\eta_{v1}}} + \frac{1}{\frac{1}{E_2}\frac{d}{dt} + \frac{1}{\eta_{v2}}} + \cdots\right]\dot{\varepsilon} \tag{D.48}$$

Thus

$$\left[\left(\frac{1}{E_1}\frac{d}{dt} + \frac{1}{\eta_{v1}}\right)\left(\frac{1}{E_2}\frac{d}{dt} + \frac{1}{\eta_{v2}}\right)\left(\frac{1}{E_3}\frac{d}{dt} + \frac{1}{\eta_{v3}}\right)\cdots\right]\sigma =$$
$$\left[\left(\frac{1}{E_2}\frac{d}{dt} + \frac{1}{\eta_{v2}}\right)\left(\frac{1}{E_3}\frac{d}{dt} + \frac{1}{\eta_{v3}}\right)\cdots + \left(\frac{1}{E_1}\frac{d}{dt} + \frac{1}{\eta_{v1}}\right)\left(\frac{1}{E_3}\frac{d}{dt} + \frac{1}{\eta_{v3}}\right)\cdots + \cdots\right]\dot{\varepsilon} \tag{D.49}$$

Next, we will consider the Kelvin models arranged in series as indicated in Figure D.23.

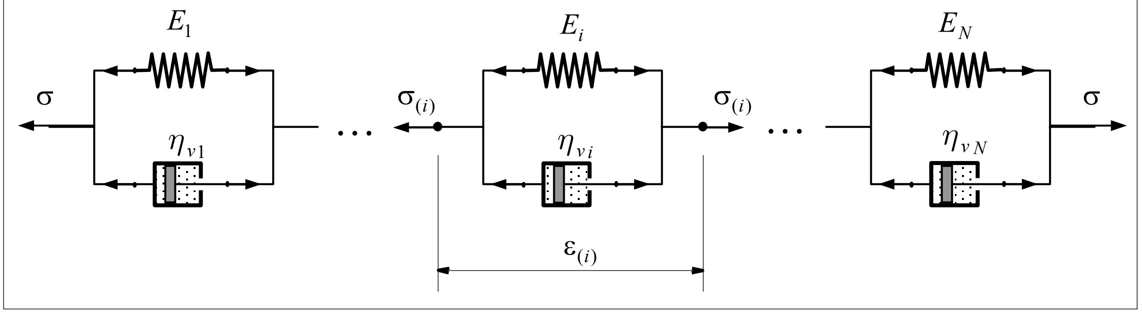

Figure D.23: Generalization of the Kelvin models in series.

The constitutive equation for the generalization of the Kelvin model described in Figure D.23, (see equation (D.22)), is given by:

$$\sigma_{(i)} = E_i \varepsilon_{(i)} + \eta_{vi} \dot{\varepsilon}_{(i)} \quad \Rightarrow \quad \sigma_{(i)} = \left(E_i + \eta_{vi} \frac{d}{dt} \right) \varepsilon_{(i)} \quad \Rightarrow \quad \varepsilon_{(i)} = \frac{\sigma_{(i)}}{\left(E_i + \eta_{vi} \frac{d}{dt} \right)} \tag{D.50}$$

The total strain for the model becomes:

$$\begin{aligned} \varepsilon &= \sum_{i=1}^{N} \varepsilon_i = \sum_{i=1}^{N} \frac{\sigma_i}{\left(E_i + \eta_{vi} \frac{d}{dt} \right)} \\ &= \frac{\sigma_{(1)}}{\left(E_1 + \eta_{v1} \frac{d}{dt} \right)} + \frac{\sigma_{(2)}}{\left(E_2 + \eta_{v2} \frac{d}{dt} \right)} + \cdots + \frac{\sigma_{(N)}}{\left(E_N + \eta_{vN} \frac{d}{dt} \right)} \end{aligned} \tag{D.51}$$

The operator $\frac{d}{dt}$ can be discarded from the denominator by multiplying both sides of the equation by $\prod_{i=1}^{a} \left(\frac{d}{dt} \eta_{vi} + E_i \right)$, with that we can obtain:

$$\begin{aligned} &\left[\left(\frac{d}{dt} \eta_{v1} + E_1 \right) \left(\frac{d}{dt} \eta_{v2} + E_2 \right) \left(\frac{d}{dt} \eta_{v3} + E_3 \right) \cdots \right] \varepsilon = \\ &\qquad \left[\left(\frac{d}{dt} \eta_{v2} + E_2 \right) \left(\frac{d}{dt} \eta_{v3} + E_3 \right) \cdots + \left(\frac{d}{dt} \eta_{v1} + E_1 \right) \left(\frac{d}{dt} \eta_{v3} + E_3 \right) \cdots + \cdots \right] \sigma \end{aligned} \tag{D.52}$$

When the stress history is prescribed, the generalization of the Kevin model is more convenient; otherwise, when the strain history is prescribed the generalization of the Maxwell model is more convenient.

D.5 Constitutive Equations in terms of Differential Operator

The constitutive equation can also be expressed in terms of differential operators as follows:

$$\left[p_0 + p_1 \frac{d}{dt} + p_2 \frac{d^2}{dt^2} + \cdots + p_a \frac{d^a}{dt^a}\right] \boldsymbol{\sigma}^{dev} = \left[q_0 + q_1 \frac{d}{dt} + q_2 \frac{d^2}{dt^2} + \cdots + q_b \frac{d^b}{dt^b}\right] \boldsymbol{\varepsilon}^{dev} \quad \text{(D.53)}$$

and

$$\left[\overline{p}_0 + \overline{p}_1 \frac{d}{dt} + \overline{p}_2 \frac{d^2}{dt^2} + \cdots + \overline{p}_a \frac{d^a}{dt^a}\right] \sigma_{kk} = \left[\overline{q}_0 + \overline{q}_1 \frac{d}{dt} + \overline{q}_2 \frac{d^2}{dt^2} + \cdots + \overline{q}_b \frac{d^b}{dt^b}\right] \varepsilon_{kk} \quad \text{(D.54)}$$

where $\sigma_{kk} = \mathrm{Tr}(\boldsymbol{\sigma})$ and $\varepsilon_{kk} = \mathrm{Tr}(\boldsymbol{\varepsilon})$. The above equations can be expressed more compactly as follows:

$$\begin{aligned} P\left(\frac{d}{dt}\right) \boldsymbol{\sigma}^{dev}(t) &= Q\left(\frac{d}{dt}\right) \boldsymbol{\varepsilon}^{dev}(t) \\ M\left(\frac{d}{dt}\right) \mathrm{Tr}(\boldsymbol{\sigma}) &= N\left(\frac{d}{dt}\right) \mathrm{Tr}(\boldsymbol{\varepsilon}) \end{aligned} \quad \text{(D.55)}$$

where the operators P, Q, M and N, with respect time, are given by:

$$P = \sum_{i=0}^{a} p_i \frac{d^i}{dt^i} \quad ; \quad Q = \sum_{i=0}^{b} q_i \frac{d^i}{dt^i} \quad ; \quad M = \sum_{i=0}^{a} \overline{p}_i \frac{d^i}{dt^i} \quad ; \quad N = \sum_{i=0}^{b} \overline{q}_i \frac{d^i}{dt^i} \quad \text{(D.56)}$$

where p_i, $\overline{p}_i$ when $(i = 1,2,\cdots,a)$, and q_i, $\overline{q}_i$, when $(i = 1,2,\cdots,b)$ are material parameters obtained experimentally.

It is possible to make analogies between viscoelastic and elastic constitutive equations. For an isotropic linear elastic material, the constitutive equations can be written in terms of Lamé constants (λ, μ), or in terms of (κ, $G = \mu$), where κ is the bulk modulus and G is the shear modulus (the transversal elastic modulus) are given by:

$$\begin{aligned} \boldsymbol{\sigma} &= \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\, \boldsymbol{\varepsilon} \\ &= \kappa \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\, \boldsymbol{\varepsilon}^{dev} \end{aligned} \qquad \begin{aligned} \sigma_{ij} &= \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \\ &= \kappa \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}^{dev} \end{aligned} \quad \text{(D.57)}$$

where $\boldsymbol{\sigma}^{dev}$ is the deviatoric part of the Cauchy stress tensor, $\boldsymbol{\varepsilon}^{dev}$ is the deviatoric part of the infinitesimal strain tensor, in which the equations $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{sph} + \boldsymbol{\sigma}^{dev} = \frac{\mathrm{Tr}(\boldsymbol{\sigma})}{3}\mathbf{1} + \boldsymbol{\sigma}^{dev}$ and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{sph} + \boldsymbol{\varepsilon}^{dev}$ hold, and $\boldsymbol{\sigma}^{sph}$ and $\boldsymbol{\varepsilon}^{sph}$ stand for the spherical parts of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, respectively.

The constitutive equation (D.57) can also be expressed by the set of equations:

$$\begin{cases} \boldsymbol{\sigma}^{dev} = 2\mu \boldsymbol{\varepsilon}^{dev} \\ \mathrm{Tr}(\boldsymbol{\sigma}) = 3\kappa \mathrm{Tr}(\boldsymbol{\varepsilon}) \end{cases} \qquad \begin{cases} \sigma_{ij}^{dev} = 2\mu \varepsilon_{ij}^{dev} \\ \sigma_{kk} = 3\kappa \varepsilon_{kk} \end{cases} \quad \text{(D.58)}$$

By comparing the equations (D.55) and (D.58) we can conclude that:

$$\mu(t) = G(t) = \frac{1}{2}\frac{Q}{P} \qquad and \qquad \kappa(t) = \frac{1}{3}\frac{N}{M} \tag{D.59}$$

Note that for viscoelastic materials the mechanical parameters depend on time. By taking into account the relationships between mechanical parameters, (see Chapter 5), the following are true $E = \frac{9\kappa G}{3\kappa + G}$ and $\nu = \frac{3\kappa - 2G}{6\kappa + 2G}$, where E is the Young's modulus, ν is the Poisson's ratio. In the same fashion, for viscoelastic materials we can define that:

$$E(t) = \frac{1}{2}\frac{3\frac{Q}{P}\frac{N}{M}}{\frac{Q}{P}+\frac{2N}{M}} = \frac{3QN}{MQ + 2PN} \quad ; \quad \nu(t) = \frac{\frac{N}{M}-\frac{Q}{P}}{\frac{2N}{M}+\frac{Q}{P}} = \frac{PN - MQ}{MQ + 2PN} \tag{D.60}$$

In general, $E(t)$ and $\nu(t)$ depend on time when we are dealing with viscoelastic materials.

Materials that behave elastically when subjected to spherical load, the volumetric operators M and N can be considered to be constants, and the equations (D.55) become:

$$\begin{aligned} &P\left(\frac{d}{dt}\right)\boldsymbol{\sigma}^{dev}(t) = Q\left(\frac{d}{dt}\right)\boldsymbol{\varepsilon}^{dev}(t) \\ &\sigma_{kk} = 3\kappa\varepsilon_{kk} \end{aligned} \tag{D.61}$$

D.6 Integral Representation of the Viscoelastic Constitutive Equations

In the previous section the constitutive equations were expressed in terms of differential operators, in this section the constitutive will expressed in terms of integral representation. This formulation is more flexible at the time of represent the actual mechanical parameters and also can be extended in order to simulate aging behavior, another advantage is that the temperature could easily be incorporated in the constitutive equations.

Creep Function

The test for creep is performed by imposing a constant stress $\sigma = \sigma_0 H(t - t_1)$ and strain $\varepsilon(t)$ is measured. For a linear material, the strain can be represented as follows:

$$\varepsilon(t) = \sigma_0 J(t) \qquad with \qquad J(t) = \frac{\varepsilon(t)}{\sigma_0} \tag{D.62}$$

where $J(t)$ is known as *Creep Function*, and is a mechanical property for viscoelastic materials.

For example, let us consider that a Maxwell material, i.e. a material which can be represented by means of the Maxwell model, has the behavior as the one described in Figure D.24.

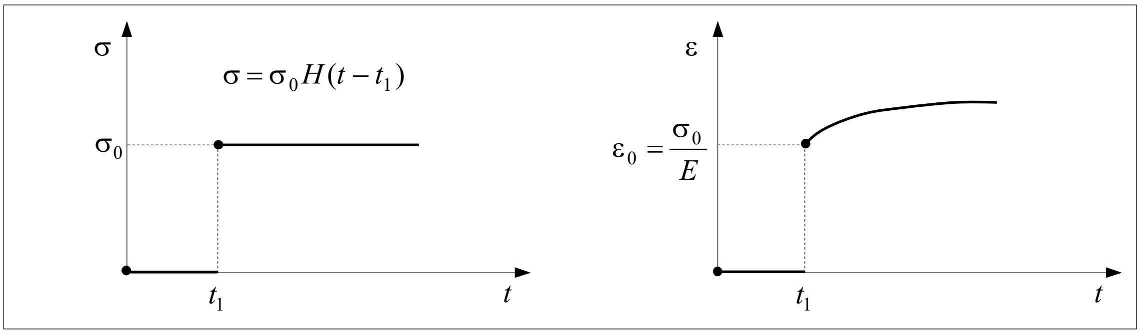

Figure D.24: Creep phenomenon under constant stress.

According to Maxwell model the equation in (D.13), $\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta_v}$ when $t > t_1$, must be fulfilled, moreover by taking into account the equation (D.62) and $\sigma = \sigma_0 H(t - t_1)$, we can obtain that when $t > t_1$ the following is true:

$$
\begin{aligned}
\dot{\varepsilon} &= \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta_v} \\
\sigma_0 \frac{\partial J(t,t_1)}{\partial t} &= \sigma_0 \frac{1}{E} \frac{\partial H(t-t_1)}{\partial t} + \frac{\sigma_0 H(t-t_1)}{\eta_v} \\
\frac{\partial J(t,t_1)}{\partial t} &= \frac{1}{E} \underbrace{\frac{\partial H(t-t_1)}{\partial t}}_{=0} + \frac{1}{\eta_v} \underbrace{H(t-t_1)}_{=1}
\end{aligned}
\tag{D.63}
$$

By integrating the above equation we can obtain:

$$
J(t,t_1) = \frac{1}{\eta_v} t + C \tag{D.64}
$$

where the constant of integration can be obtained by means of initial conditions:

$$
\varepsilon(t) = \sigma_0 J(t,t_1) \quad when \quad t = t_1 \quad \Rightarrow \quad J_0 = \frac{\varepsilon_0}{\sigma_0} = \frac{1}{E} \tag{D.65}
$$

Then, the creep function for the Maxwell model is given by:

$$
J(t,t_1) = \frac{1}{\eta_v}(t - t_1) + \frac{1}{E} \quad when \quad t \geq t_1 \tag{D.66}
$$

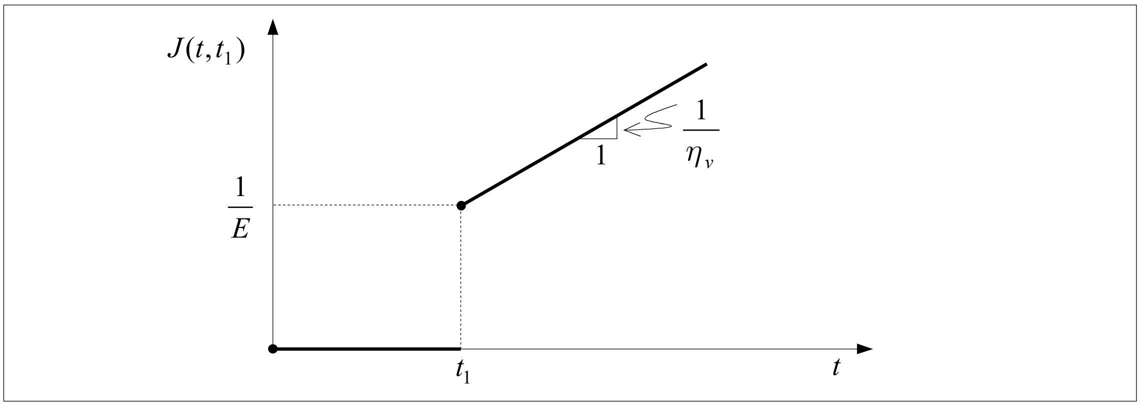

Figure D.25: Creep function for the Maxwell model.

Relaxation Function

The test for relaxation is performed by imposing a constant strain $\varepsilon = \varepsilon_0 H(t)$ and the stress $\sigma(t)$ is measured, in which $\sigma(t)$ is a function of time. If the material has a linear behavior the stress can be represented as follows:

$$\sigma(t) = \varepsilon_0 E(t) \qquad with \qquad E(t) = \frac{\sigma(t)}{\varepsilon_0} \tag{D.67}$$

where $E(t)$ is the *Relaxation Function*, which is a material mechanical property. For example, for a Maxwell material in which the equation $\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta_v}$ holds, and by taking into account the equation in (D.67), we can obtain the relaxation function as follows:

$$E(t) = E \exp^{\frac{-Et}{\eta_v}} \tag{D.68}$$

Boltzmann Superposition Principle. Integral Representation

If the constant stress is applied at time $t = \xi_1$, then $\sigma(t) = \sigma_{(1)} H(t - \xi_1)$ holds and the strain (creep) associated to this stress state is given by:

$$\varepsilon(t) = \sigma_{(1)} J(t - \xi_1) H(t - \xi_1) \tag{D.69}$$

The above equation represents the same results as (D.62) but displaced in time by ξ_1.

If a linear viscoelastic material is subjected to the initial stress σ_0 at time $t = 0$, and at time $t = \xi_1$ it is applied the stress $\sigma_{(1)}$ the total strain for time greater than ξ_1 is given by the sum of the strain for each state separately, (see Figure D.26). This is known as the *Boltzmann superposition principle*.

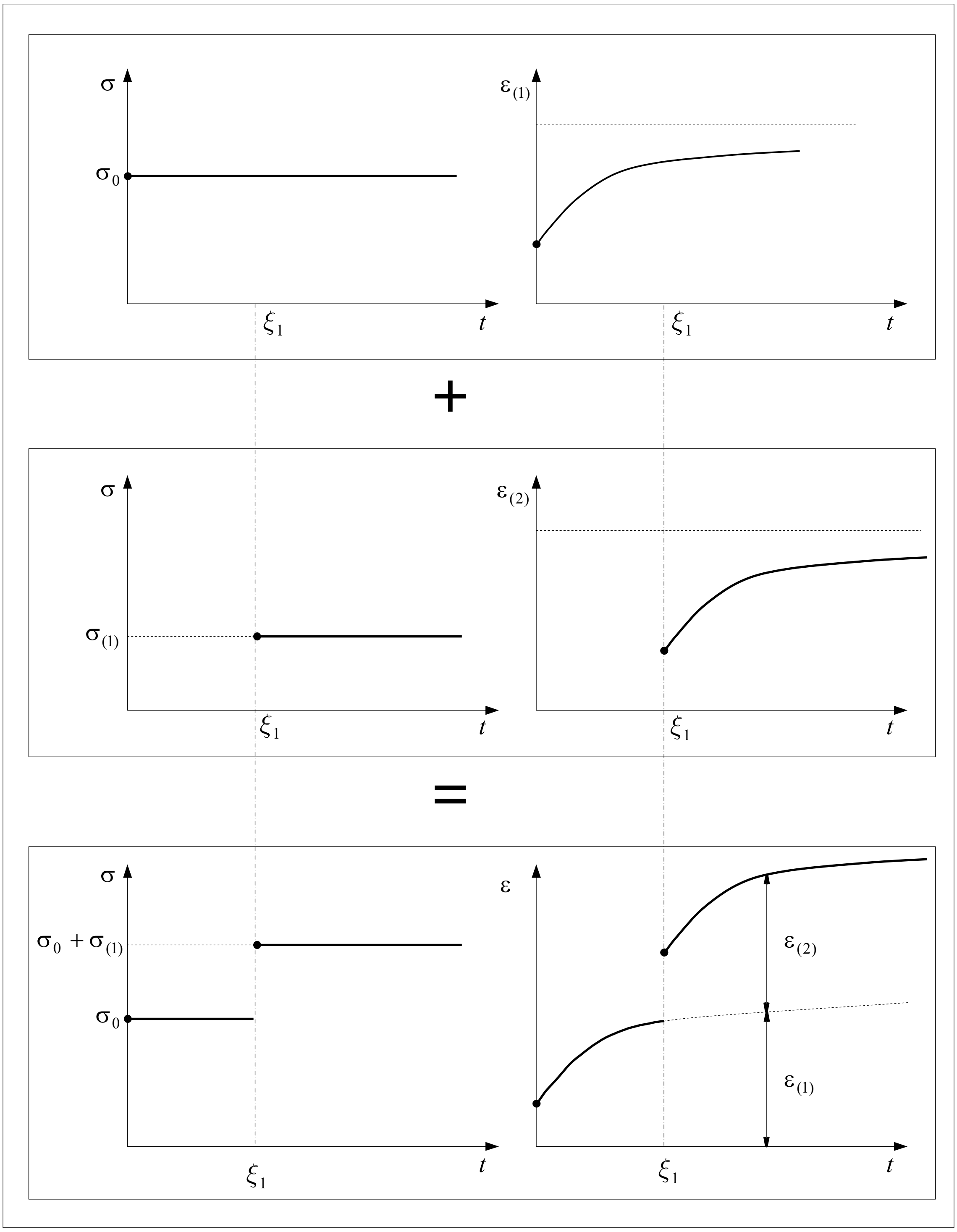

Figure D.26: Boltzmann superposition principle.

Consider now that the stress $\sigma(t)$ varies continuously over time as indicated in Figure D.27.

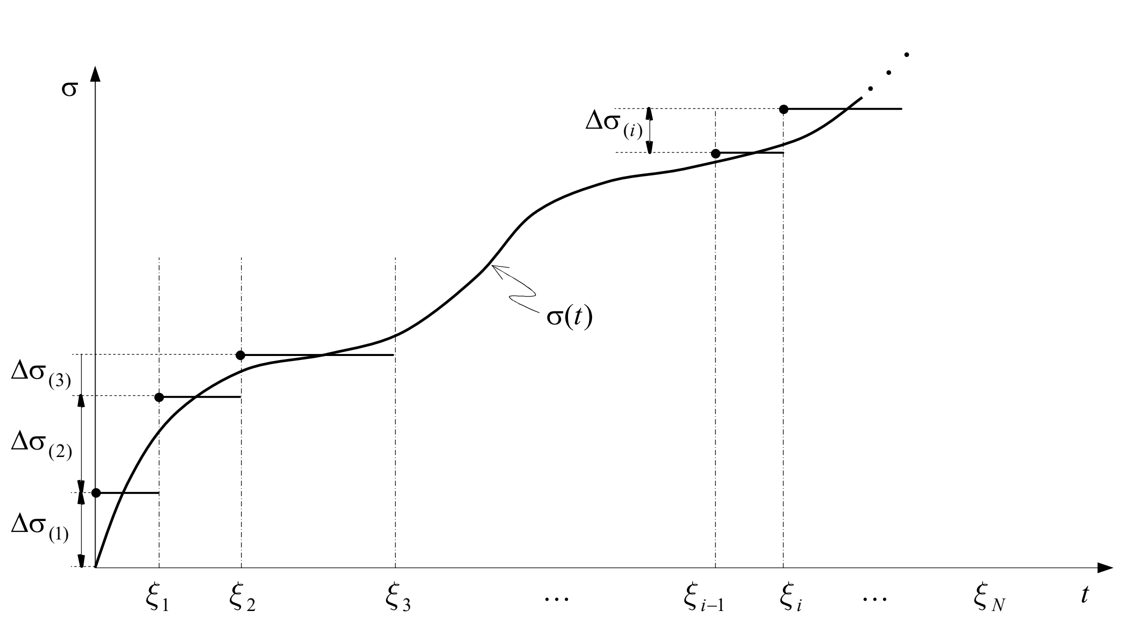

Figure D.27: Continuous stress function over time.

The continuous stress function $\sigma(t)$ can be approached the summation of the intervals of time in which the stress is considered constant for each interval, (see Figure D.27), i.e.:

$$\sigma(t) \cong \sum_{i=1}^{N} \Delta\sigma_i H(t-\xi_i) \tag{D.70}$$

The smaller the interval the better the approximation for $\sigma(t)$ is.

The Boltzmann superposition principle can be applied in order to obtain the total strain as follows:

$$\varepsilon(t) \cong \sum_{i=1}^{N} \varepsilon_i(t-\xi_i) = \sum_{i=1}^{r} \sigma_i J(t-\xi_i) H(t-\xi_i) \tag{D.71}$$

If the time interval tends to infinite the total strain can be expressed by means of integral representation as follows:

$$\varepsilon(t) = \int_0^t J(t-\xi) H(t-\xi) d\sigma(\xi) \tag{D.72}$$

Integrals in each the integrand depend on time are called Stieltjes integrals. If the stress function is differentiable and since ξ is smaller than t, the function $H(t-\xi)$ will the unity into the integration range, then the integral (D.72) reduces to:

$$\boxed{\varepsilon(t) = \int_0^t J(t-\xi) \frac{\partial\sigma(\xi)}{\partial\xi} d\xi = \int_0^t J(t-\xi)\dot{\sigma}(\xi) d\xi} \quad \textit{Integral representation of the creep phenomenon} \tag{D.73}$$

in which it was considered $d\sigma(\xi) = \frac{\partial\sigma(\xi)}{\partial\xi} d\xi$, since $\sigma(\xi)$ is differentiable when $\xi > 0$.

The integral representation (D.73) can be applied in order to describe (predict) the strain due to creep phenomenon under a stress history represented by creep function $J(t)$.

An alternative way for (D.73) can be obtained by applying integration by parts in (D.73) and by considering $u = J(t-\xi)$ and $v' = \frac{\partial \sigma}{\partial \xi} d\xi$ we can obtain:

$$\varepsilon(t) = J(t-\xi)\sigma(\xi)\Big|_0^t - \int_0^t \frac{dJ(t-\xi)}{d\xi}\sigma(\xi)d\xi = \sigma(t)J(0) - \int_0^t \frac{dJ(t-\xi)}{d\xi}\sigma(\xi)d\xi \qquad \text{(D.74)}$$

The creep function $J(t)$ can be split into an elastic (J_0), independent of time, and a part which depends on time ($\varphi(t)$), with that we can obtain:

$$\varepsilon(t) = J_0\sigma(t) - \int_0^t \varphi(t-\xi)\frac{\partial \sigma(\xi)}{\partial \xi} d\xi \qquad \text{(D.75)}$$

The equation (D.73), with $J(t)$ obtained by means of experiments in the laboratory, can be applied to predict the stress by mans of prescribed strain history. Nevertheless, the problem could be not easy to solve since it requires solving the integral equation instead of integrating directly.

The Boltzmann superposition principle can also be applied in order to obtain the reverse problem, i.e. for several strain states and the stress function can be obtained as the summation of the intervals of time in which the stress is considered constant for each interval in terms of time:

$$\boxed{\sigma(t) = \int_0^t E(t-\xi)\frac{\partial \varepsilon(\xi)}{\partial \xi} d\xi} \quad \textit{Integral representation of the relaxation phenomenon} \qquad \text{(D.76)}$$

The function $E(t)$ can be split into an elastic (E_0), independent of time, and a part which depends on time ($\psi(t)$), with that we can obtain:

$$\sigma(t) = E_0\varepsilon(t) - \int_0^t \psi(t-\xi)\frac{\partial \varepsilon(\xi)}{\partial \xi} d\xi \qquad \text{(D.77)}$$

Relationship between the Creep Function and the Relaxation Function

Since the creep and relaxation phenomena are two aspects related to the same viscoelastic material, these phenomena must be related to each other, so if true, know one phenomenon it is possible to obtain the other phenomenon.

By applying the Laplace Transformation in (D.73) and (D.76) we can obtain the equations in terms of s:

$$\left.\begin{array}{lll} \hat{\varepsilon}(s) = s\hat{J}(s)\hat{\sigma}(s) & \Rightarrow & \dfrac{\hat{\sigma}(s)}{\hat{\varepsilon}(s)} = s\hat{J}(s) \\ \hat{\sigma}(s) = s\hat{E}(s)\hat{\varepsilon}(s) & \Rightarrow & \dfrac{\hat{\sigma}(s)}{\hat{\varepsilon}(s)} = s\hat{E}(s) \end{array}\right\} \Rightarrow \quad \hat{J}(s)\hat{E}(s) = \frac{1}{s^2} \qquad \text{(D.78)}$$

By applying the inverse Laplace transformation:

$$\int_0^t J(t-\xi)E(\xi)d\xi = t \tag{D.79}$$

or

$$\int_0^t E(t-\xi)J(\xi)d\xi = t \tag{D.80}$$

The equation (D.78), (D.79) and (D.80) define the relationship between the Creep function $J(t)$ and the Relaxation function $E(t)$ for linear viscoelastic materials.

D.7 Integral Representation for Three-Dimensional Problems

The extension of the one-dimensional problem to three-dimensional problem would be intuitive and immediate according to the equation (D.73) and (D.76). Nevertheless, we will apply the fundamental principle of Continuum Mechanics in order to obtain the Integral Representation for three-dimensional viscoelastic problem, (Christensen (1982)).

The complete motion history of the continuum is expressed by means of the motaition equations:

$$x_i(\xi) = x_i(X_j, \xi) \quad ; \quad -\infty < \xi \le t \tag{D.81}$$

where ξ is time variable and t is the current time.

Recall that a functional is an operator that provides the current value of the variable by taking into account its whole history, so the stress function is represented as follows:

$$\sigma_{ij}(t) = \mathop{\Xi}_{s=0}^{\infty}\left(\varepsilon_{kl}(t-s), \varepsilon_{kl}(t)\right) \tag{D.82}$$

where $\mathop{\Xi}\limits_{s=0}^{\infty}$ is a real value functional, which provides the current value of $\sigma_{ij}(t)$ by taking into account the whole history of $\varepsilon_{ij}(t)$, for $-\infty < t < \infty$.

If the strain history is continuous and the functional is linear, we can apply the Riesz integral representation in order to rewrite the functional (D.82) as a Stieltjes integral:

$$\sigma_{ij}(t) = \int_0^{\infty} \varepsilon_{kl}(t-s) d\mathbb{E}_{ijkl}(s) \tag{D.83}$$

where $\mathbb{E}$ is a fourth-order tensor which contains the relaxation functions such as:

$$\mathbb{E}(t) = \mathbf{0} \qquad when \qquad -\infty < t < 0 \tag{D.84}$$

Due to the symmetries of the stress and strain the tensor $\mathbb{E}$ presents minor symmetry, we can also prove that $\mathbb{E}$ presents major symmetry, so:

$$\begin{gathered} \mathbb{E}_{ijkl}(t) = \mathbb{E}_{klij}(t) \\ \mathbb{E}_{ijkl}(t) = \mathbb{E}_{jikl}(t) = \mathbb{E}_{ijlk}(t) \end{gathered} \tag{D.85}$$

The above anisotropic tensor presents 21 independent relaxation functions.

Let us assume that $\varepsilon_{ij}(t)=0$ when $t<0$ and also that $\mathbb{E}(t)$ and its first derivative are continuous functions into the range $0\le t<\infty$, with that the equation (D.83) can be writen as follows:

$$\sigma_{ij}(t)=\mathbb{E}_{ijkl}(0)\varepsilon_{kl}(t)+\int_0^t \varepsilon_{kl}(t-s)\frac{\partial \mathbb{E}_{ijkl}(s)}{\partial s}ds \tag{D.86}$$

By changing the nomenclature such as $\xi=t-s$ and by integrating by parts the above equation we can obtain an alternative form for the constitutive equations as follows:

$$\sigma_{ij}(t)=\int_0^t \mathbb{E}_{ijkl}(t-\xi)\frac{\partial \varepsilon_{kl}(\xi)}{\partial \xi}d\xi \qquad \boldsymbol{\sigma}(t)=\int_0^t \mathbb{E}(t-\xi):\dot{\boldsymbol{\varepsilon}}(\xi)d\xi \tag{D.87}$$

Integral representation for relaxation in 3D

In the case in which the function ε_{kl} has a jump at time $t=0$, we can obtain that:

$$\sigma_{ij}(t)=\mathbb{E}_{ijkl}(t)\varepsilon_{kl}(0)+\int_0^t \mathbb{E}_{ijkl}(t-\xi)\frac{\partial \varepsilon_{kl}(\xi)}{\partial \xi}d\xi \tag{D.88}$$

An alternative way of expressing the stress-strain relation can be obtained by doing the same approach as done previously, but now reversing the roles between stress ansd strain, and we obtain the integral representation of the creep phenomenon for three-dimensional problems:

$$\varepsilon_{ij}(t)=\int_{-\infty}^t \mathbb{J}_{ijkl}(t-\xi)\frac{\partial \sigma_{kl}(\xi)}{\partial \xi}d\xi \qquad \boldsymbol{\varepsilon}(t)=\int_{-\infty}^t \mathbb{J}(t-\xi):\dot{\boldsymbol{\sigma}}(\xi)d\xi \tag{D.89}$$

Representación integral del fenómeno de fluencia en 3D

where $\mathbb{J}(t)$ is a fourth-order tensor that contains the 21 independent creep functions, and the tensor $\mathbb{J}(t)$ also presents minor symmetry, i.e. $\mathbb{J}_{ijkl}=\mathbb{J}_{jikl}=\mathbb{J}_{ijlk}$ and $\mathbb{J}(t)=\mathbf{0}$ when $\infty<t<0$. Moreover, $\mathbb{J}(t)$ and its first derivative are continuous functions into the range $0\le t<\infty$.

Recall from the linear elasticity theory that the strain-stress relationship is given by:

$$\varepsilon_{ij}=\mathbb{C}_{ijkl}\sigma_{kl} \xrightarrow{\text{Rate of Change}} \dot{\varepsilon}_{ij}=\mathbb{C}_{ijkl}\dot{\sigma}_{kl} \tag{D.90}$$

where $\mathbb{C}$ is the fourth-order tensor (the elasticity tensor). By means of analogy with viscoelastic materials in which:

$$\varepsilon_{ij}(t)=\int_0^t \mathbb{J}_{ijkl}(t-\xi)\dot{\sigma}_{kl}(\xi)d\xi \tag{D.91}$$

where $\mathbb{J}$ is the fourth-order tensor (the relaxation tensor). In the most general case in which the material is completely anisotropic $\mathbb{J}$ has 21 independent relaxation functions. When we are dealing with isotropic material $\mathbb{J}$ has 2 independent relaxation functions.

For isotropic linear elastic materials the constitutive equations can be expressed by means of a spherical and deviatoric part as follows:

$$\sigma_{ij} = \sigma_{ij}^{sph} + \sigma_{ij}^{dev} = \kappa\, \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}^{dev} \quad \Rightarrow \quad \dot{\sigma}_{ij} = \kappa\, \dot{\varepsilon}_{kk}\delta_{ij} + 2\mu\dot{\varepsilon}_{ij}^{dev} \tag{D.92}$$

where δ_{ij} is the Kronecker delta.

In the same fashion we can establish the constitutive equations for viscoelastic materials as follows:

$$\begin{aligned}
\sigma_{ij}^{dev}(t) &= 2\int_0^t \mu(t-\xi)\frac{\partial \varepsilon_{ij}^{dev}(\xi)}{\partial \xi}d\xi \quad ; \quad \sigma_{ij}^{sph}(t) = \delta_{ij}\int_0^t \kappa(t-\xi)\frac{\partial \varepsilon_{kk}(\xi)}{\partial \xi}d\xi \\
\boldsymbol{\sigma}^{dev}(t) &= 2\int_0^t \mu(t-\xi)\frac{\partial \boldsymbol{\varepsilon}^{dev}(\xi)}{\partial \xi}d\xi \quad ; \quad \boldsymbol{\sigma}^{sph}(t) = \left(\int_0^t \kappa(t-\xi)\frac{\partial \mathrm{Tr}[\boldsymbol{\varepsilon}(\xi)]}{\partial \xi}d\xi\right)\mathbf{1}
\end{aligned} \tag{D.93}$$

where $\mu(t)$ is the shear (transversal) relaxation function, and $\kappa(t)$ is the bulk relaxation function.

The constitutive equations for isotropic linear elastic material can also be written in terms of Lamé constants (λ, μ):

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{D.94}$$

In the same fashion, the constitutive equations for isotropic linear viscoelastic materials can be written as follows:

$$\sigma_{ij}(t) = \delta_{ij}\left[\lambda\varepsilon_{kk} - \int_0^t \psi_1(t-\xi)\frac{\partial \varepsilon_{kk}(\xi)}{\partial \xi}d\xi\right] + 2\mu\varepsilon_{ij}(t) - \int_0^t \psi_2(t-\xi)\frac{\partial \varepsilon_{ij}(\xi)}{\partial \xi}d\xi \tag{D.95}$$

where λ and μ are time independent constants that describes the stress-strain relation, ψ_1 and ψ_2 are the relaxation functions for ε_{ij} and ε_{kk} respectively.

The reverse form of the constitutive equation (D.94) is given by:

$$\varepsilon_{ij} = \frac{-\lambda}{2\mu(2\mu+3\lambda)}\sigma_{kk}\delta_{ij} + \frac{1}{2\mu}\sigma_{ij} \tag{D.96}$$

In the same fashion, the function $\varepsilon_{ij}(t)$ can be write as follows

$$\varepsilon_{ij}(t) = \delta_{ij}\left[a_0\sigma_{kk}(t) + \int_0^t \varphi_1(t-\xi)\frac{\partial \sigma_{kk}(\xi)}{\partial \xi}d\xi\right] + b_0\sigma_{ij}(t) + \int_0^t \varphi_2(t-\xi)\frac{\partial \sigma_{ij}(\xi)}{\partial \xi}d\xi \tag{D.97}$$

where $a_0 = \dfrac{-\lambda}{2\mu(2\mu+3\lambda)}$ and $b_0 = \dfrac{1}{2\mu}$ are material parameters, φ_1 and φ_2 are the creep functions.

D.8 Initial Boundary Value Problem. The Elastic-Viscoelastic Correspondence Principle

Let us consider a three-dimensional body $\mathcal{B}$ (deformed configuration) which has a volume V and mass density ρ. Let S be the boundary of $\mathcal{B}$ and $\hat{\mathbf{n}}$ be the outward unit normal to the surface S. Then, we shall consider that the body is moving under the action of body forces $\vec{\mathbf{b}}(\vec{x})$ and under traction forces $\vec{\mathbf{t}}^*(\vec{x})$ (prescribed value). The boundary consists of a part $S_{\mathbf{u}}$ in which the displacements are prescribed and a part $S_{\boldsymbol{\sigma}}$ where the traction vector is prescribed (surface force), such that $\overline{S_{\mathbf{u}} \cup S_{\boldsymbol{\sigma}}} = S$ and $S_{\mathbf{u}} \cap S_{\boldsymbol{\sigma}} = \varnothing$, (see Figure D.28).

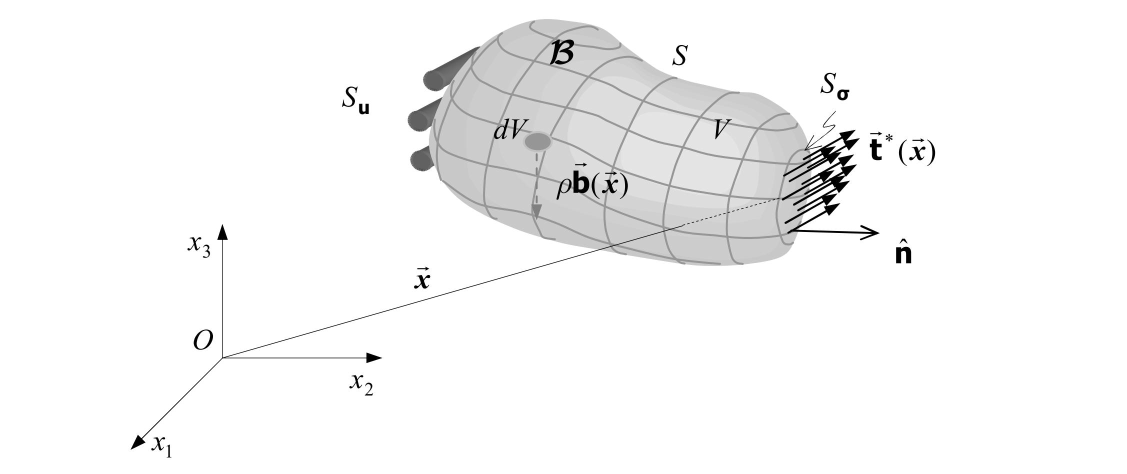

Figure D.28: Continuum medium in motion.

Recall that for isotropic linear elastic materials are governed by the following set of equations:

1) Equations of motion:

$$\nabla \cdot \boldsymbol{\sigma}(\vec{x},t) + \rho\vec{\mathbf{b}}(\vec{x},t) = \rho \frac{\partial^2 \vec{\mathbf{u}}(\vec{x},t)}{\partial t^2} \quad ; \quad \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \mathrm{b}_i = \rho \frac{\partial^2 \mathrm{u}_i}{\partial t^2} \tag{D.98}$$

2) Constitutive equations for stress:

$$\boldsymbol{\sigma} = \lambda \mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \quad ; \quad \sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{D.99}$$

3) Kinematic equations:

$$\boldsymbol{\varepsilon}(\vec{x},t) = \nabla^{sym}\vec{\mathbf{u}}(\vec{x},t) \quad ; \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \mathrm{u}_i}{\partial x_j} + \frac{\partial \mathrm{u}_j}{\partial x_i}\right) \tag{D.100}$$

And the initial and boundary conditions are:

Boundary condition in displacement on $S_{\mathbf{u}}$:

$$\vec{\mathbf{u}}(\vec{x},t) = \vec{\mathbf{u}}^*(\vec{x},t) \quad ; \quad \mathrm{u}_i(\vec{x},t) = \mathrm{u}_i^*(\vec{x},t) \tag{D.101}$$

Boundary conditions in stress on $S_{\boldsymbol{\sigma}}$:

$$\boldsymbol{\sigma}(\vec{x},t) \cdot \hat{\mathbf{n}} = \vec{\mathbf{t}}^*(\vec{x},\hat{\mathbf{n}},t) \quad ; \quad \sigma_{jk}\mathrm{n}_k = \mathrm{t}_j^*(\vec{x},t) \tag{D.102}$$

And the initial conditions at time $t = 0$:

$$\begin{aligned} &\vec{\mathbf{u}}(\vec{x},t) = \vec{\mathbf{u}}_0 \\ &\left.\frac{\partial \vec{\mathbf{u}}_0(\vec{x},t)}{\partial t}\right|_{t=0} = \dot{\vec{\mathbf{u}}}_0(\vec{x},t) = \vec{v}_0(\vec{x}) \end{aligned} \tag{D.103}$$

In order to extend the above formulation to isotropic linear viscoelastic materials we can apply the *Correspondence Principle*. This principle arises in an analogous way between the governing equations for elasticity problem and the Laplace transformation related to the viscoelastic constitutive equations. So, by considering the isothermal material we can establish:

Linear Elastic Problem	Linear Viscoelastic Problem
Equations of Motion	
$\nabla\cdot\boldsymbol{\sigma}(\vec{x},t)+\rho\vec{\mathbf{b}}(\vec{x},t)=\rho\dfrac{\partial^2\vec{\mathbf{u}}(\vec{x},t)}{\partial t^2}$	$\nabla\cdot\overline{\boldsymbol{\sigma}}(\vec{x},t)+\rho\bar{\vec{\mathbf{b}}}(\vec{x},t)=\rho\dfrac{\partial^2\bar{\vec{\mathbf{u}}}(\vec{x},t)}{\partial t^2}$
Kinematic Equations	
$\boldsymbol{\varepsilon}(\vec{x},t)=\nabla^{sym}\vec{\mathbf{u}}$	$\overline{\boldsymbol{\varepsilon}}(\vec{x},t)=\nabla^{sym}\bar{\vec{\mathbf{u}}}$
Constitutive Equations	
$\boldsymbol{\sigma}=\lambda\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon}$ *or* $\begin{cases}\boldsymbol{\sigma}^{dev}=2\mu\boldsymbol{\varepsilon}^{dev}\\ \mathrm{Tr}(\boldsymbol{\sigma})=3\kappa\mathrm{Tr}(\boldsymbol{\varepsilon})\end{cases}$	$\begin{cases}\boldsymbol{\sigma}^{dev}(t)=2\displaystyle\int_0^t\mu(t-\xi)\frac{\partial\boldsymbol{\varepsilon}^{dev}(\xi)}{\partial\xi}d\xi\\ \boldsymbol{\sigma}^{sph}(t)=\left(\displaystyle\int_0^t\kappa(t-\xi)\frac{\partial\mathrm{Tr}[\boldsymbol{\varepsilon}(\xi)]}{\partial\xi}d\xi\right)\mathbf{1}\end{cases}$ (D.104)
Boundary Conditions	
$\vec{\mathbf{u}}(\vec{x},t)=\vec{\mathbf{u}}^*(\vec{x},t)$ on $S_{\mathbf{u}}$	$\bar{\vec{\mathbf{u}}}(\vec{x},t)=\bar{\vec{\mathbf{u}}}^*(\vec{x},t)$ on $S_{\mathbf{u}}$
$\boldsymbol{\sigma}(\vec{x},t)\cdot\hat{\boldsymbol{n}}=\vec{\mathbf{t}}^*(\vec{x},\hat{\boldsymbol{n}},t)$ on $S_{\boldsymbol{\sigma}}$	$\overline{\boldsymbol{\sigma}}(\vec{x},t)\cdot\overline{\boldsymbol{n}}=\bar{\vec{\mathbf{t}}}^*(\vec{x},\hat{\boldsymbol{n}},t)$

where the variables with bar, i.e. $(\bar{\bullet})$, indicates the Laplace transformation according to the definition:

$$\bar{f}(x_i,s)=\int_0^\infty f(x_i,t)\exp^{-st}dt \tag{D.105}$$

Viscoelasticity References

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