Annex C

Incremental-Iterative Strategy Solution

C.1 Introduction

For a better understanding of the behavior of structures and materials, it has increased the demand to formulate algorithms that are able to realistically simulate the complete behavior of the structure/material. Among one of the applications we can mention the simulation of a structure to its complete destruction. In this way it allows us to design more efficiently the structure in order to face a disaster (earthquakes, explosions, etc.). To achieve this objective we must take into account non-linearity behavior. Basically we can highlight two types of non-linearity:

- Material Non-Linearity;
- Geometric Non-Linearity.

The material non-linearity appears when the stress-strain relationship is non-linear. The geometric non-linearity occurs when deformed configuration (current state) has great influence in the outcome. When we are dealing with the Finite Element Formulation the strain field $\boldsymbol{\varepsilon}(\vec{x})$ is related to the nodal displacements $\{\boldsymbol{u}\}^{(e)}$ by means the matrix which contains the derivatives of the shape functions $[\boldsymbol{B}]$, which in small deformation regime is only a function of initial geometric parameters, $\boldsymbol{\varepsilon}(\vec{x}) = [\boldsymbol{B}]\{\boldsymbol{u}\}^{(e)}$. In the case of finite deformation regime the matrix $[\boldsymbol{B}]$ is also a function of the displacement field, *i.e.* $\boldsymbol{\varepsilon}(\vec{x}) = [\boldsymbol{B}(\boldsymbol{u})]\{\boldsymbol{u}\}^{(e)}$. And as consequence the stiffness matrix is also a function of the displacement field.

In general, the geometric non-linearity is a consequence of the large displacement that undergoes the structure. Then, the measure of strain adopted must be able to capture the real displacement of the structure. Several measures of strain have been established, e.g. Green-Lagrange strain tensor, Almansi strain tensor, Logarithmic strain tensor, etc.

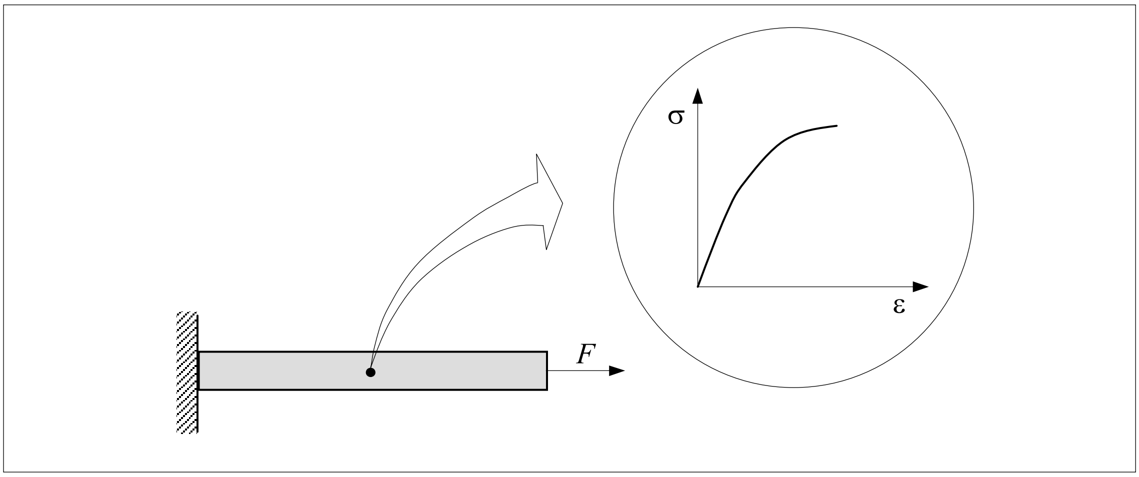

Figure C.1: Material non-linearity.

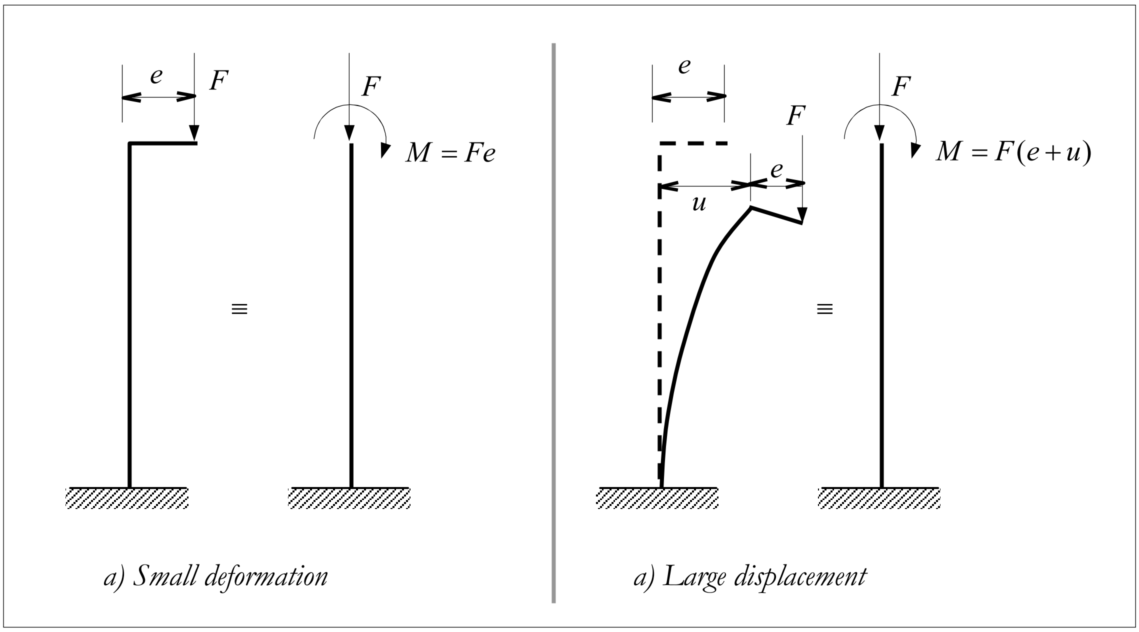

Figure C.2: Boundary conditions non-linearity.

C.1.1 Solution Strategies

To achieve the previous objectives, various solution techniques have been proposed. The choice of numerical algorithm depends on the given problem. For example, if the structure response is represented by a curve Force vs. Displacement, (see Figure C.3), the aim is to obtain the complete curve of the graph in question. As we will see later, we can use a strategy that is force increment (*force control*). But there may be a point, for instance, point A of the graph, where the force control procedure will get a no desired point as solution (point F) or even a divergence of the solution. Another strategy used is through incremental displacement (*displacement control*), which can also have undesirable solution if we are at point B and to a further increment in displacement we reach the point D of the graph missing all curve from B to D. Another strategy which can be employed is a combination of the two above methodologies, i.e. force control and displacement control simultaneously, which is known as *Arc-length control*.

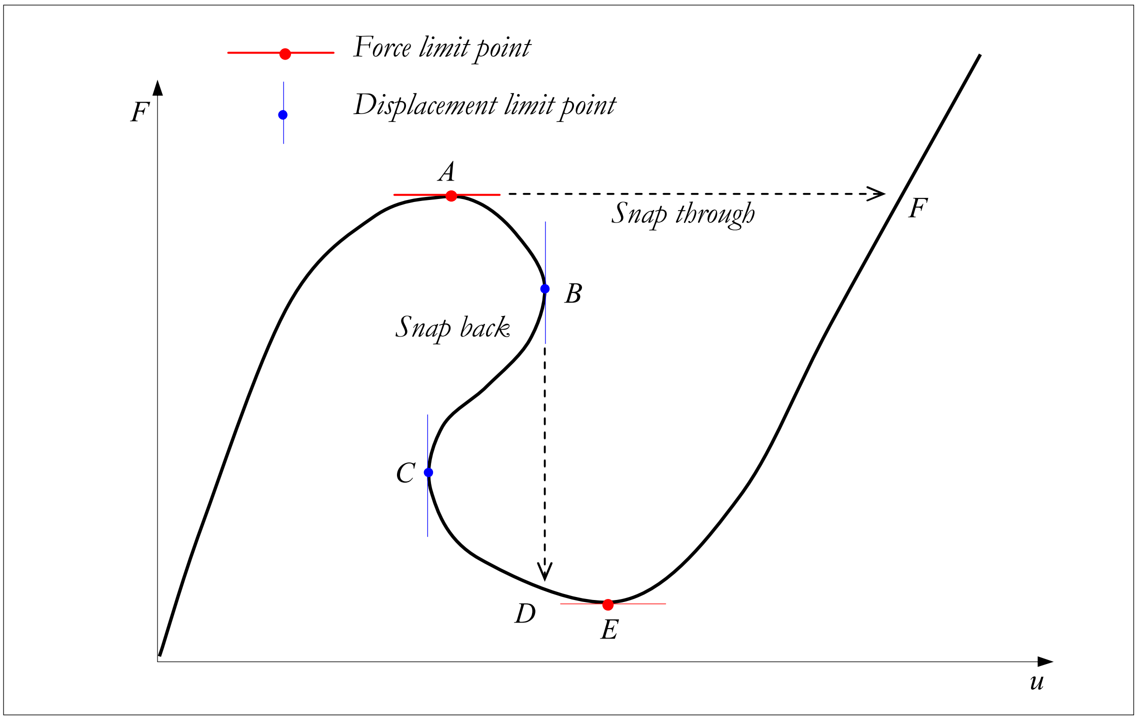

Figure C.3: Force-displacement curve.

Sometimes when we are using an incremental strategy it has an error associated with it or even the solution can diverge, (see Figure C.4). To overcome this drawback we must use an incremental-iterative scheme. Among the iterative methods we can mention the Newton-Raphson's method.

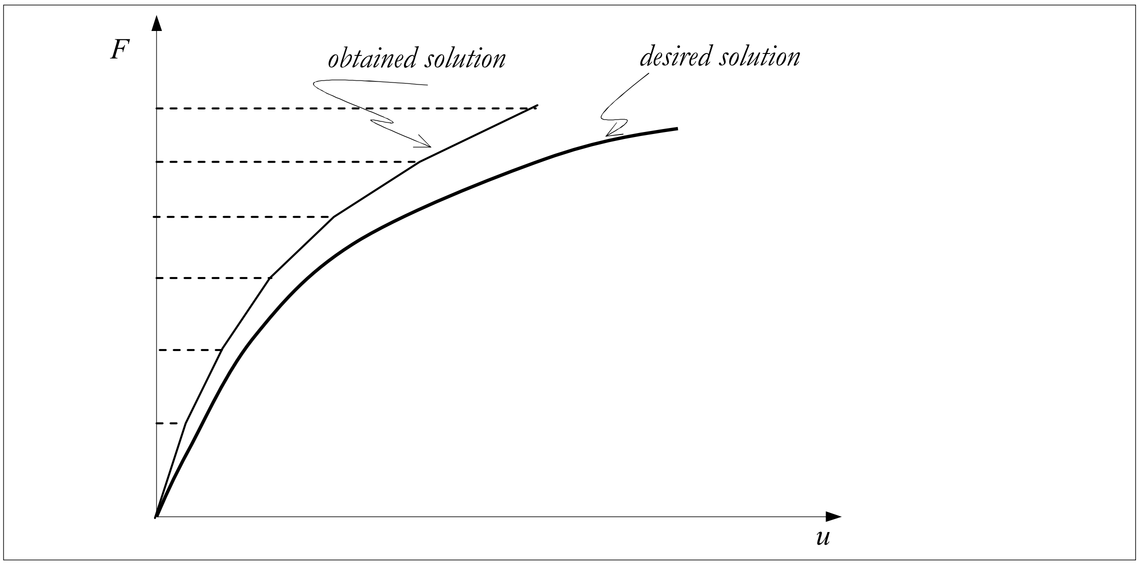

Figure C.4: Diverting of the solution

C.2 Total Potential Energy

The total potential energy (Π) of an elastic system is composed by two parts, namely:

- Internal potential energy (strain energy potential (U^{int}));
- External potential energy (U^{ext}):

$$\Pi(z) = U^{int}(z) - U^{ext}(z) \tag{C.1}$$

If the equation (C.1) represents the stationary condition for the total potential energy, it can be shown that the value of z is an extreme of $\Pi(z)$. This is the *Principle of stationary of the total potential energy*. Then, if we looking for the extremes (maximums and minimums) of the function, (see Figure C.5), the following must hold:

$$\frac{d\Pi(z)}{dz} = 0 \tag{C.2}$$

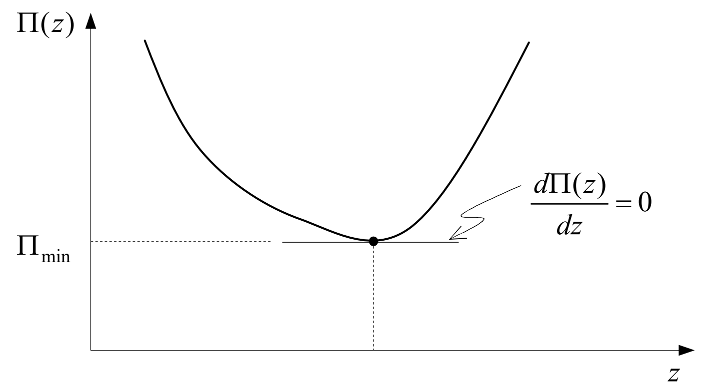

Figure C.5: Minimum of a function

Let us suppose a one-dimensional case in which $U^{ext}(u) = Wz$. With that the condition (C.2) becomes:

$$\begin{aligned}&\frac{d\Pi(z)}{dz} = \frac{dU^{int}(z)}{dz} - \frac{dU^{ext}(z)}{dz} = \frac{dU^{int}(z)}{dz} - W = B(z) - W = 0\\ &\Rightarrow B(z) = W\end{aligned} \tag{C.3}$$

where we have considered $B(z) = \frac{dU^{int}(z)}{dz}$. We can rewrite the above equation as follows:

$$W = \left[\frac{B(z)}{z}\right] z = K^{Sec} z \tag{C.4}$$

where K^{Sec} is the secant of the curve $W \times z$ at the point z, (see Figure C.6).

Let us suppose that for a give increment Δz we have:

$$B(z + \Delta z) = W + \Delta W \tag{C.5}$$

In addition, by means of Taylor series the equation $B(z+\Delta z) = B(z) + \frac{\partial B(z)}{\partial z}\Delta z + \dots$ holds, in which we have only considered linear terms. Then, the equation (C.5) can be rewritten as follows:

$$
\begin{aligned}
& B(z+\Delta z) = B(z) + \frac{\partial B(z)}{\partial z}\Delta z = W + \Delta W \\
& \Rightarrow \underbrace{B(z) - W}_{=0} + \frac{\partial B(z)}{\partial z}\Delta z = \Delta W \\
& \Rightarrow \frac{\partial B(z)}{\partial z}\Delta z = \Delta W \\
& \Rightarrow K^T \Delta z = \Delta W
\end{aligned} \tag{C.6}
$$

where $K^{tan} = \frac{\partial B(z)}{\partial z} = \frac{\partial^2 U^{int}(z)}{\partial z^2}$ is the tangent of the curve $W \times z$ at the point z, (see Figure C.6).

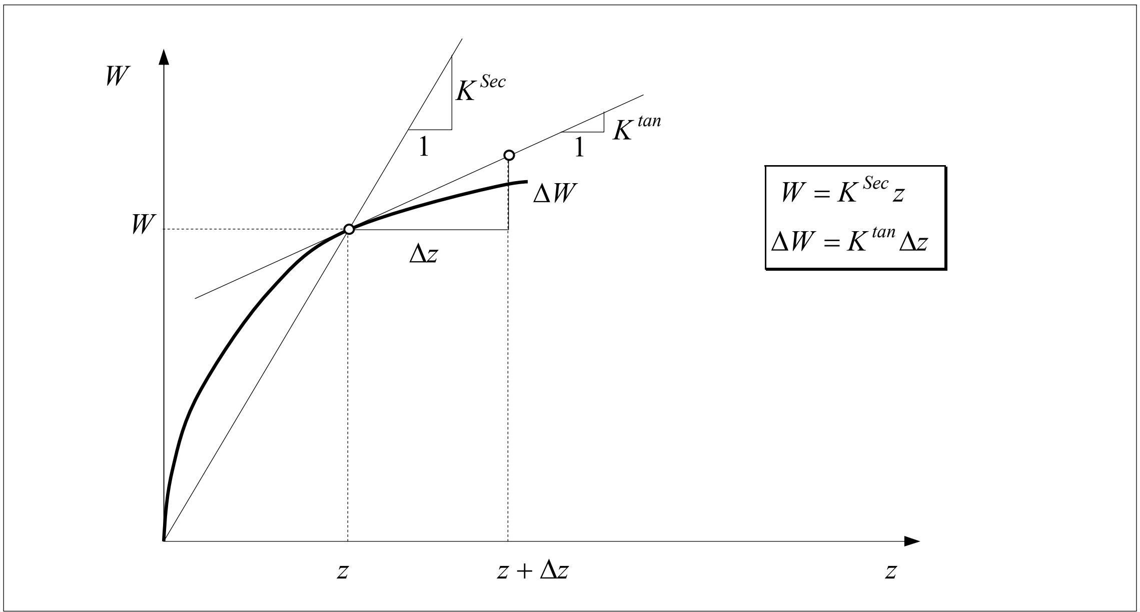

Figure C.6: Tangent *vs.* Secant.

C.2.1 Extension to n Dimensions

Suppose that the total potential energy is a function of n variables, i.e. $\Pi = \Pi(u_i)$ for $i = 1,2,3,\cdots,n$. Then, we can obtain that

$$
\begin{aligned}
& \frac{d\Pi(\boldsymbol{u})}{du_i} = \frac{dU^{int}(\boldsymbol{u})}{du_i} - \frac{dU^{ext}(\boldsymbol{u})}{du_i} = \frac{dU^{int}(\boldsymbol{u})}{du_i} - F_i = B_i(\boldsymbol{u}) - F_i = 0 \\
& \Rightarrow B_i(\boldsymbol{u}) = F_i
\end{aligned} \tag{C.7}
$$

With that we can defined the Secant stiffness matrix $\boldsymbol{K}^{Sec}$ as follows:

$$
F_i = \left[\frac{B_i(\boldsymbol{u})}{u_j}\right] u_j = K_{ij}^{Sec} u_j \tag{C.8}
$$

Note that we are using indicial notation. Similarly to obtain the equations in (C.6) we can define the Tangent stiffness matrix $\boldsymbol{K}^{tan}$ as follows:

$$
\frac{\partial B_i(\boldsymbol{u})}{\partial u_j}\Delta u_j = \Delta F_i \qquad \Rightarrow \qquad K_{ij}^{tan}\Delta u_j = \Delta F_i \tag{C.9}
$$

where

$$K_{ij}^{tan} = \frac{\partial B_i(\boldsymbol{u})}{\partial u_j} = \frac{\partial^2 U^{int}(\boldsymbol{u})}{\partial u_i \partial u_j} \quad \textit{Tangent stiffness matrix} \tag{C.10}$$

C.3 Example with 1 Degree-of-Freedom

In this subsection we will adopt the example in the book Crisfield (1991), (see Figure C.7), where A stands for cross-section area of the bar and E represents the Young's modulus.

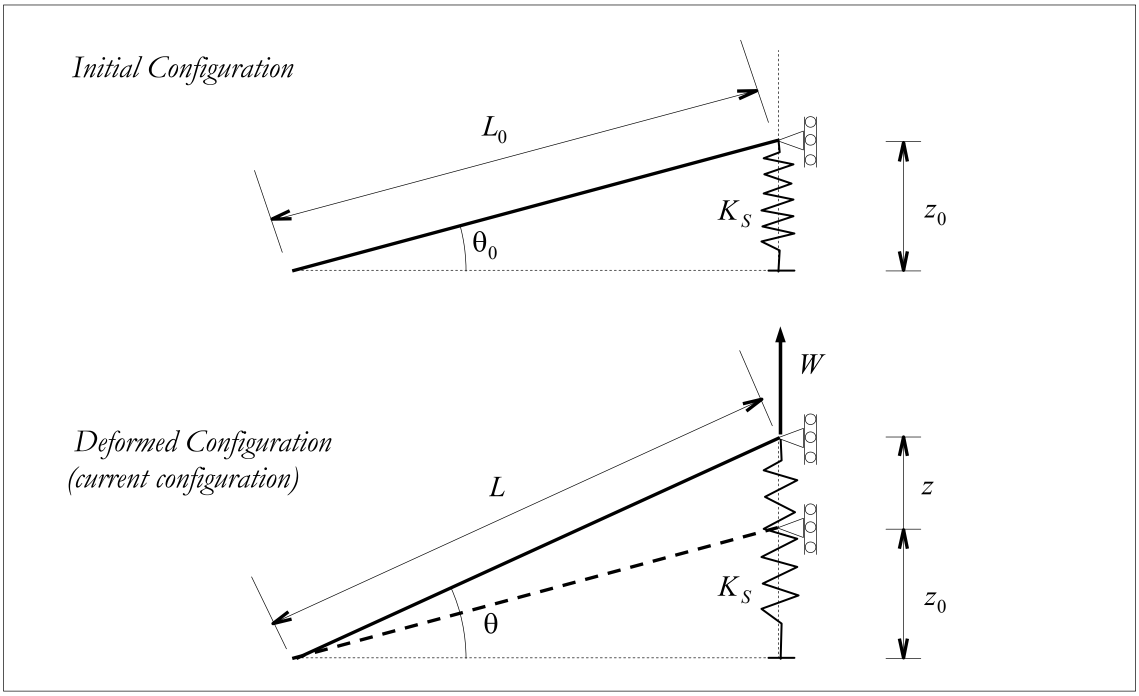

Figure C.7: Example with 1 degree-of-freedom, (Crisfield(1991)).

Making the vertical equilibrium at the node in which the force W is applied, (see Figure C.8), we can obtain:

$$W = N \sin\theta + K_S z = N\frac{(z_0 + z)}{L} + K_S z \cong N\frac{(z_0 + z)}{L_0} + K_S z \tag{C.11}$$

where N is the axial force of the bar and K_S is the spring coefficient. In Crisfield (1991) more detail about this formulation is provided.

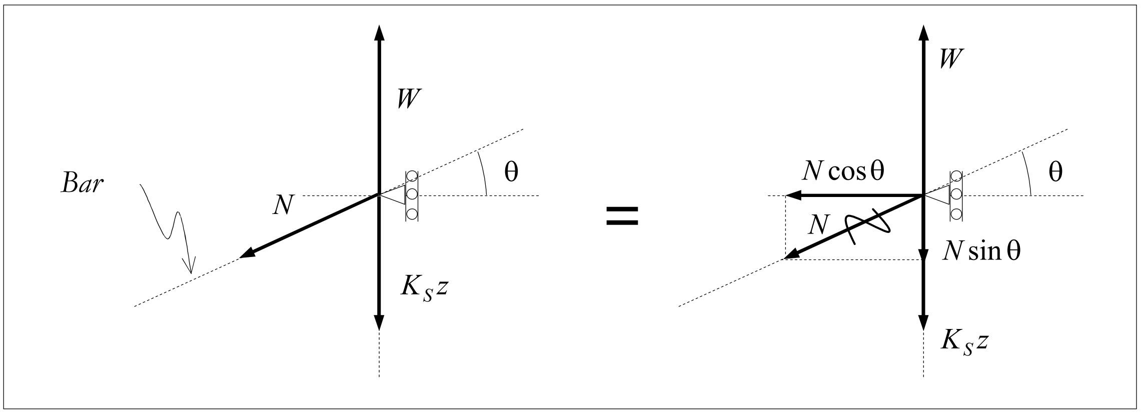

Figure C.8: Equilibrium at the node.

Additionally we can state that:

$$N = \sigma A = EA\varepsilon = EA\left[\left(\frac{z_0}{L_0}\right)\left(\frac{z}{L_0}\right) + \frac{1}{2}\left(\frac{z}{L_0}\right)^2\right] \tag{C.12}$$

and

$$W = W(z) = \frac{EA}{L_0^3}\left[z_0^2 z + \tfrac{3}{2} z_0 z^2 + \tfrac{1}{2} z^3\right] + K_S z \tag{C.13}$$

For a given value of z we can calculate the tangent of the curve $W = W(z)$ as follows:

$$K^{tan} = \frac{dW}{dz} = \frac{d}{dz}\left[\frac{EA}{L_0^3}\left(z_0^2 z + \tfrac{3}{2} z_0 z^2 + \tfrac{1}{2} z^3\right) + K_S z\right] = \frac{EA}{L_0^3}\left(z_0^2 + 3z_0 z + \tfrac{3}{2} z^2\right) + K_S \tag{C.14}$$

We can also express K^{tan} in terms of axial force as follows:

$$\begin{aligned} K^{tan} &= \frac{dW}{dz} = \frac{(z_0 + z)}{L_0}\frac{dN}{dz} + \frac{N}{L_0} + K_S \\ &= \frac{EA}{L_0}\left(\frac{z_0 + z}{L_0}\right)^2 + \frac{N}{L_0} + K_S \\ &= \frac{EA}{L_0^3}(z_0 + z)^2 + \frac{N}{L_0} + K_S \end{aligned} \tag{C.15}$$

Next we will adopt values for $z < 0$ in order to draw the curve $W \times z$, (see Figure C.9), which represents the exact value of the function $W = W(z)$.

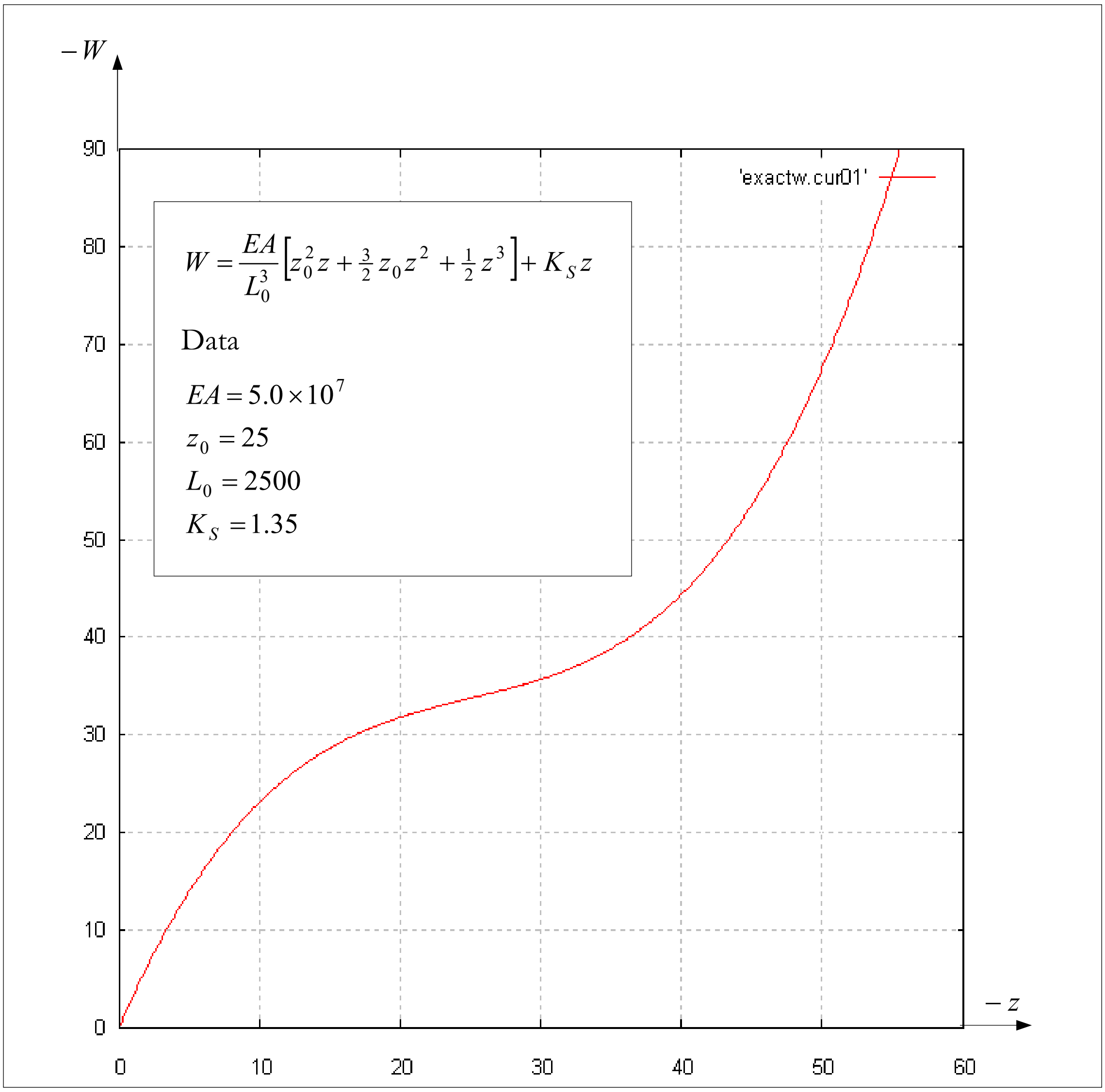

Figure C.9: Function $W(z)$.

Now let us assume that we do not know the function $W(z)$, but instead we know its first derivative $K^{tan}=\frac{dW}{dz}$. In order to obtain the function $W(z)$ we can adopt the Euler's method, (see Annex A). With this method we can control the force $W(z)$ throughout increment of ΔW or we can control the displacement z throughout increments of Δz, (see Figure C.10).

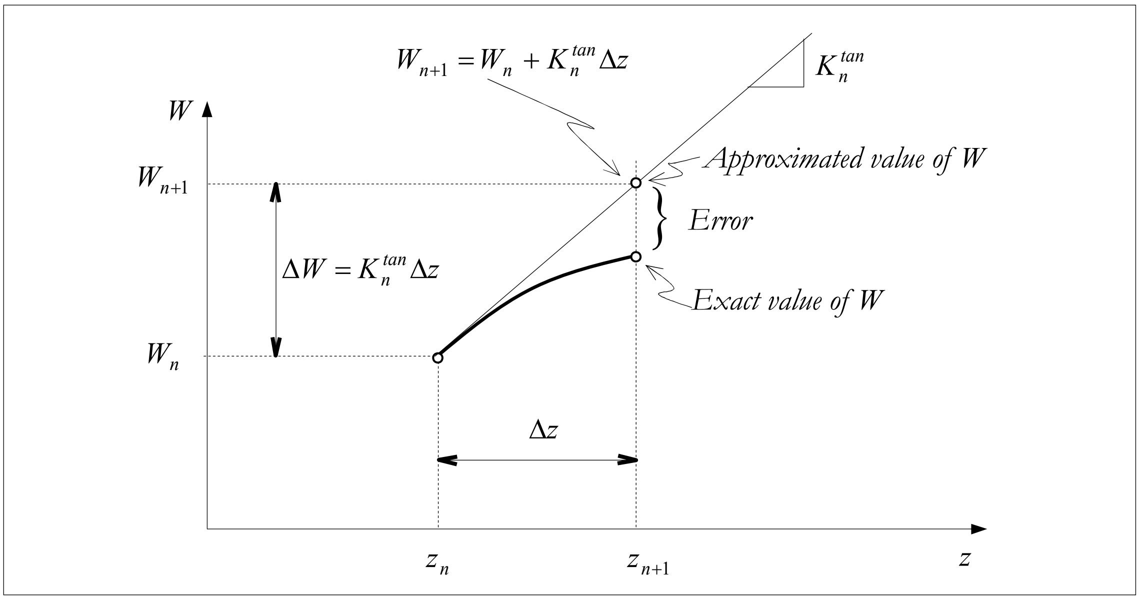

Figure C.10: Euler's method (Implicit method).

For increments of Δz we can apply:

$$K_n^{tan} = \frac{W_{n+1} - W_n}{\Delta z} \quad \Rightarrow \quad W_{n+1} = W_n + K_n^{tan} \Delta z \tag{C.16}$$

And for increments of ΔW we can apply:

$$K_n^{tan} = \frac{W_{n+1} - W_n}{\Delta z} = \frac{\Delta W}{\Delta z} \quad \Rightarrow \quad \Delta z = \left(K_n^{tan}\right)^{-1} \Delta W \tag{C.17}$$

We will apply these two methodologies in order to obtain numerically the curve $W \times z$. For the first case we will impose a displacement increment of $\Delta z = 5.0$ and for the second case an increment in force equal to $\Delta W = 7.0$, (see Figure C.11). If we want a more accurate solution we will need to adopt very small increments, but this procedure could cause a time-consuming from a computational point of view when we are dealing with several degrees-of-freedom. To overcome this drawback we can adopt the incremental-iterative scheme, in which even with big increments we can obtain good results.

The procedure in FORTRAN code can be found at the link: NON_LINEAR1D.FOR, (starting at label 20 for Displacement control and starting at label 30 for Load control).

When we are using force control the solution diverges when the force starts to decrease. To illustrate this behavior let us consider same example, but changing the value of K_S by $K_S = -0.35$, (see Figure C.9). In this situation we can verify that when we are adopting the force control the curve diverge, (see Figure C.12). For this example we have adopt a force increment equal to $\Delta W = 2$.

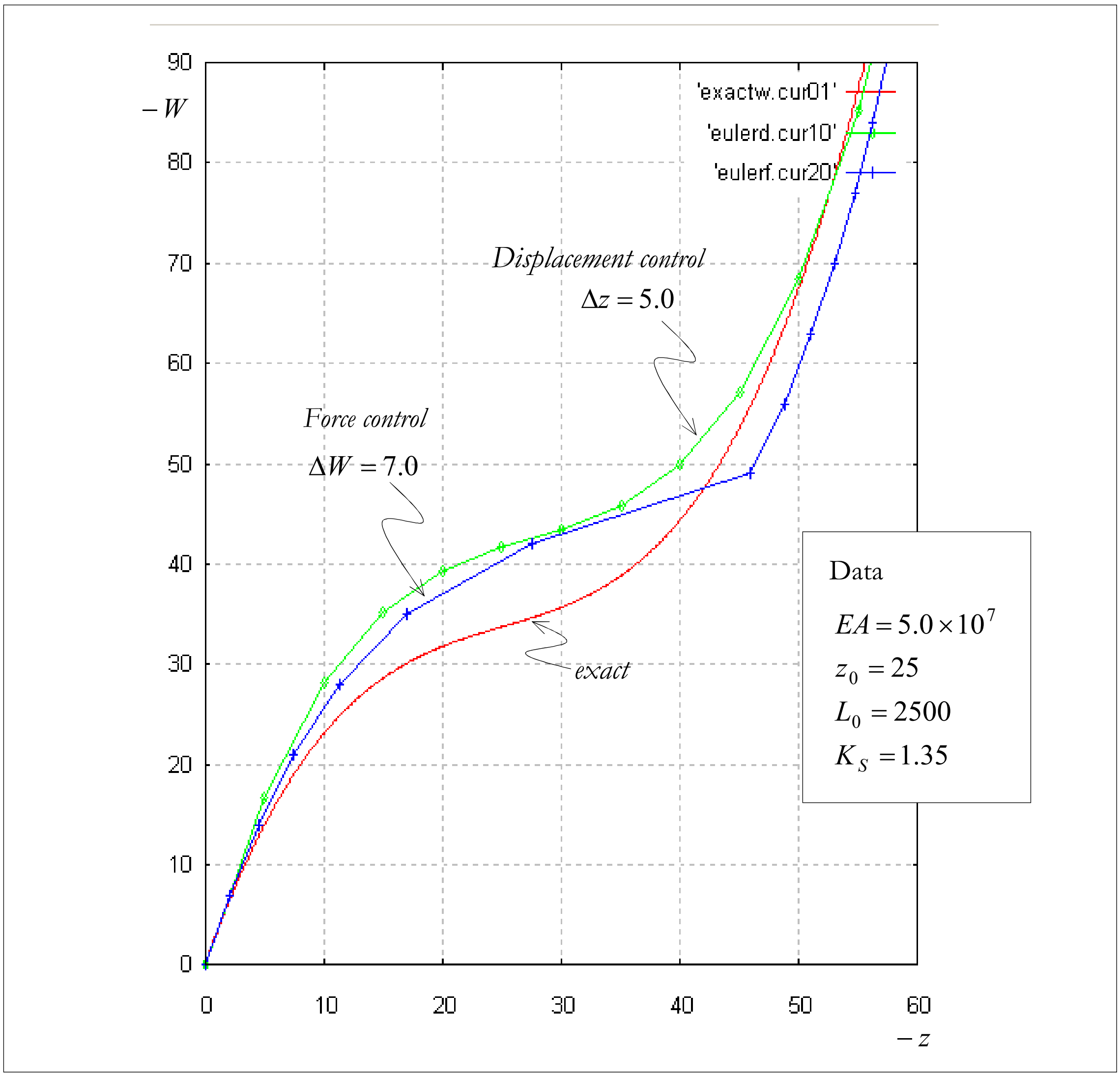

Figure C.11: Incremental solution.

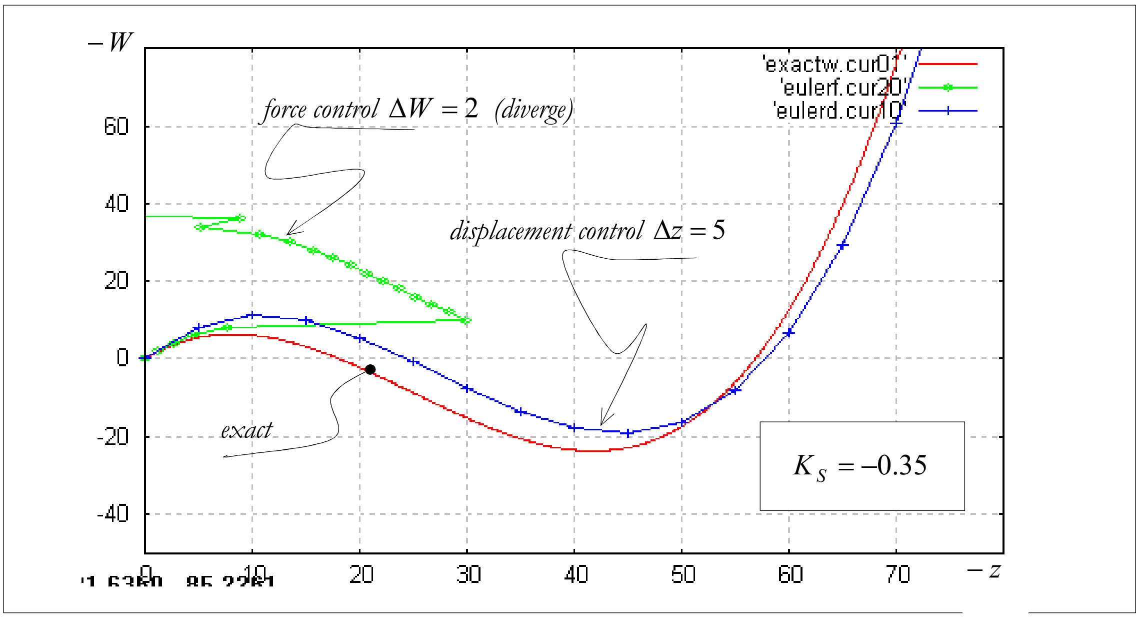

Figure C.12: Incremental solution.

C.3.1 Total Potential Energy

For the example given by Section C.3 the total potential energy is given by:

$$\Pi(z) = U^{int}(z) - U^{ext}(z) = \left(\frac{1}{2}\int_0^{L_0} \frac{N^2}{EA} d\overline{x} + \frac{1}{2} K_S z^2 \right) - (Wz) \tag{C.18}$$

which is the same as:

$$\Pi(z) = \frac{L_0}{2EA} N^2 + \frac{1}{2} K_S z^2 - Wz \tag{C.19}$$

Note that the axial force depends on z, i.e. $N = N(z)$.

By applying the Principle of stationary of the total potential energy we can obtain:

$$\begin{aligned} &\frac{d\Pi(z)}{dz} = \frac{d}{dz}\left(\frac{L_0}{2EA} N^2 + \frac{1}{2} K_S z^2 - Wz \right) = 0 \\ &\Rightarrow \underbrace{\frac{L_0}{EA} N \frac{dN}{dz} + K_S z}_{=\frac{dU^{int}}{dz}} - \underbrace{W}_{=\frac{dU^{ext}}{dz}} = 0 \end{aligned} \tag{C.20}$$

By considering the equation in (C.12) we can obtain:

$$\frac{dN}{dz} = \frac{EA}{L_0^2}\left[z_0 + z \right] \tag{C.21}$$

And by substituting the equation (C.21) into (C.20) we can obtain:

$$\begin{aligned} &\frac{L_0}{EA} N \frac{dN}{dz} + K_S z - W = 0 \quad \Rightarrow \quad \frac{L_0}{EA} N \frac{EA}{L_0^2}\left[z_0 + z \right] + K_S z - W = 0 \\ &\Rightarrow \frac{N}{L_0}\left[z_0 + z \right] + K_S z - W = 0 \\ &\Rightarrow W = \frac{N}{L_0}\left[z_0 + z \right] + K_S z \end{aligned} \tag{C.22}$$

As expected, the above equation is the same as the one given by the equation in (C.11). The tangent matrix can be obtained by means of:

$$\begin{aligned} \frac{d^2 U^{int}(z)}{dz^2} &= \frac{d}{dz}\left[\frac{N}{L_0}(z_0 + z) + K_S z \right] \\ &= \frac{1}{L_0} \frac{dN}{dz}(z_0 + z) + \frac{N}{L_0} + K_S \\ &= \frac{1}{L_0} \frac{EA}{L_0^2}\left[z_0 + z \right](z_0 + z) + \frac{N}{L_0} + K_S \end{aligned} \tag{C.23}$$

which results in:

$$\frac{d^2 U^{int}(z)}{dz^2} = K^{tan} = \frac{EA}{L_0^3}(z_0 + z)^2 + \frac{N}{L_0} + K_S \tag{C.24}$$

which matches the equation in (C.15).

Next, we will draw the curve $\Pi \times z$. To do so, let us adopt the following values for the load W, namely:

$$\begin{aligned} z_1 &= -2.5 \quad \Rightarrow \quad W_1 = -7.65 \\ z_2 &= -4.0 \quad \Rightarrow \quad W_2 = -11.5824 \\ z_3 &= -5.0 \quad \Rightarrow \quad W_3 = -13.95 \end{aligned} \tag{C.25}$$

where we have used the data of Figure C.9. By means of the total potential energy given by equation (C.19) and with fixed values of W_1, W_2 and W_3 we can obtain, respectively, three curves Π_1, Π_2 and Π_3, (see Figure C.13). As we can verify, for each potential function the extreme (minimum) corresponds to the displacement associated with W_i.

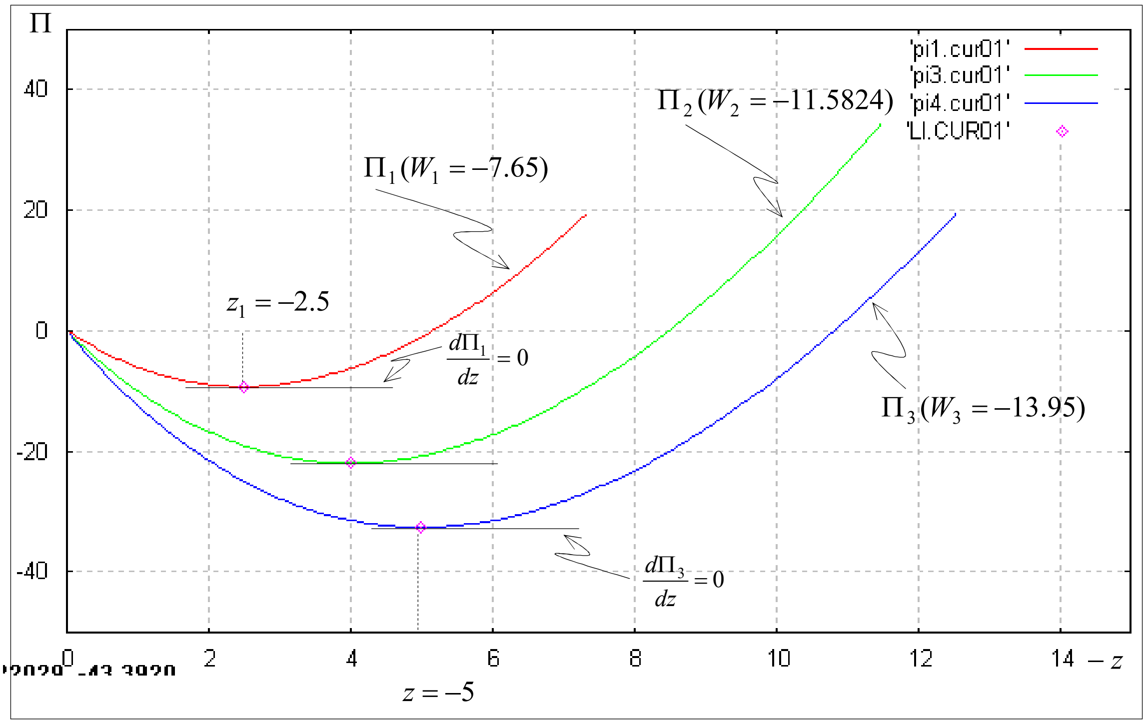

Figure C.13: Total potential ($K_S = 1.35$).

Let us now turn our attention to the example in Figure C.12, where we have considered that $K_S = -0.35$. And we draw the total potential when the load is equal to $W = -2$, (see Figure C.15). If we look at Figure C.14 we can verify that for the value $W = -2$ we have three solutions, namely $z_1 = -1.34047$, $z_2 = -16.24045$ and $z_3 = -57.41908$, (see Figure C.14), whose values are the roots of the cubic equation:

$$\frac{EA}{L_0^3}\left[z_0^2 z + \tfrac{3}{2} z_0 z^2 + \tfrac{1}{2} z^3\right] + K_S z - W = 0$$

These roots correspond exactly to the extremes of Π, i.e. when $\frac{d\Pi}{dz} = 0$, (see Figure C.15). We also emphasize that at z_1 and at z_3 the tangent matrix is positive definite, while at z_2 the tangent matrix is negative definite, in this case indicating that the solution diverges for force increment, but not for an increment of displacement (displacement control).

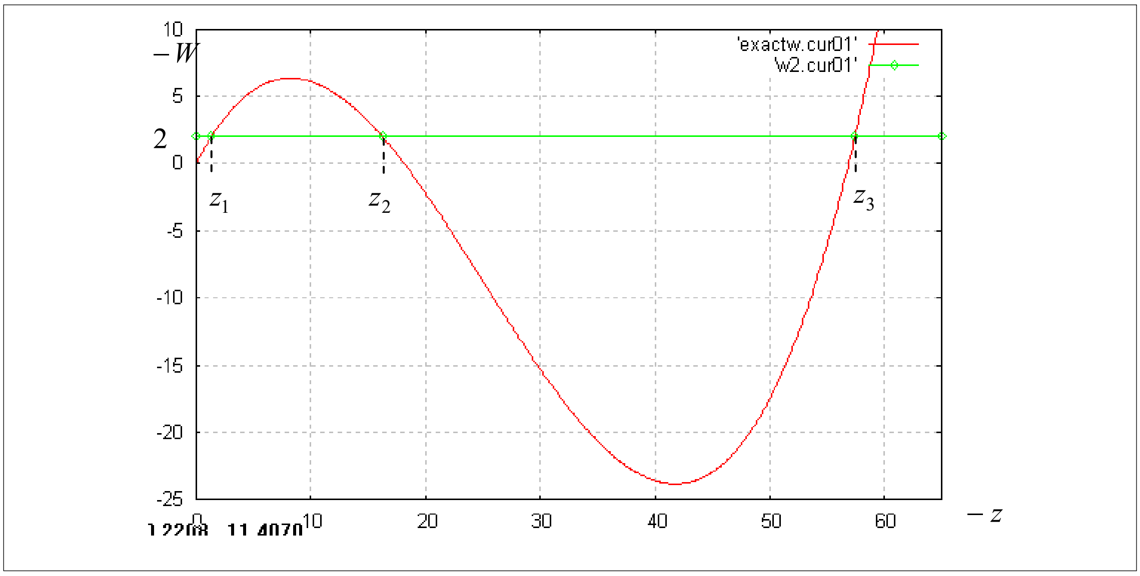

Figure C.14: Function W when $K_S = -0.35$.

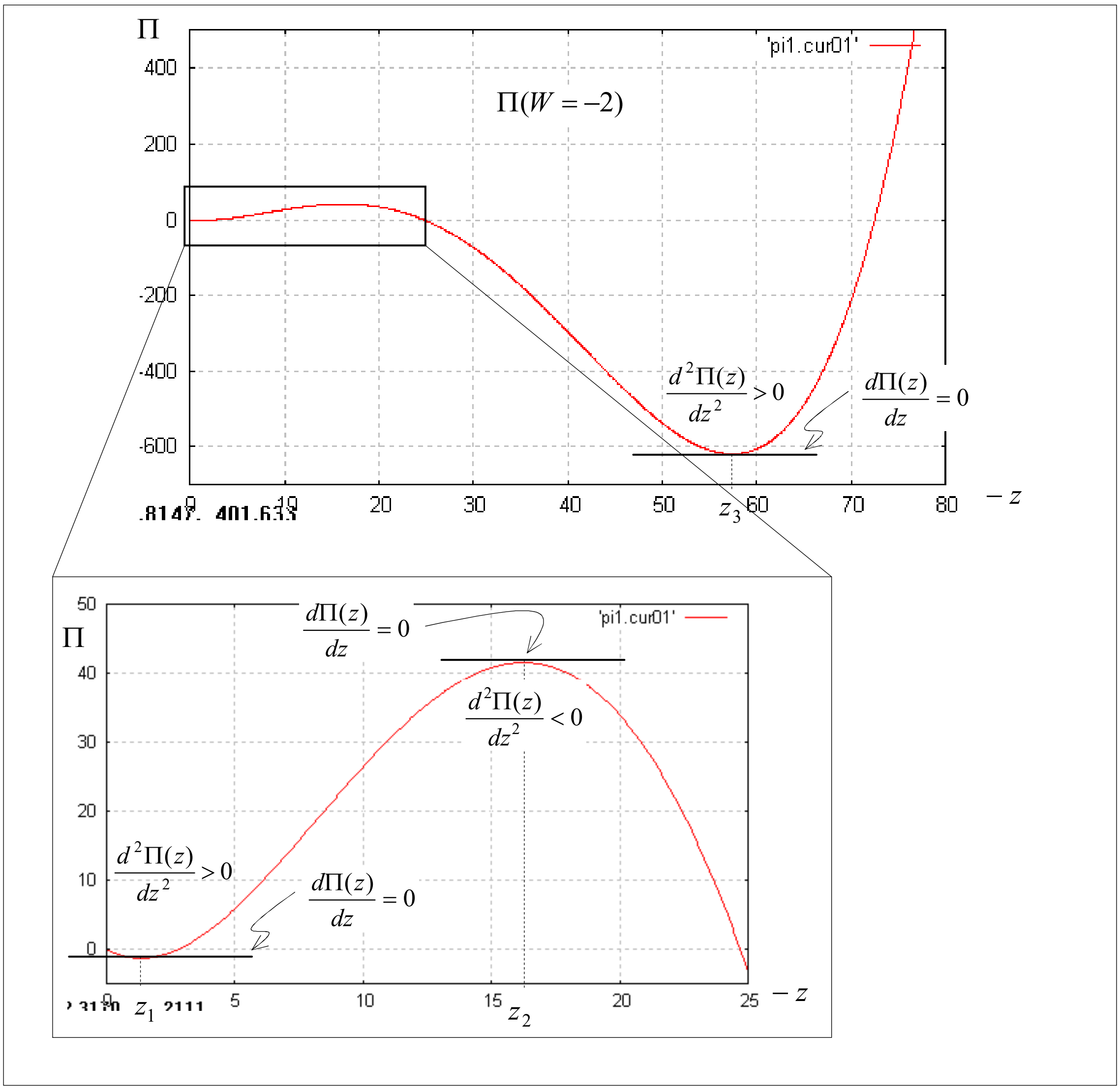

Figure C.15: Total potential $K_S = -0.35$.

C.4 Analyzing the Total Potential Energy

For the next analysis let us consider that we are at the equilibrium point and we add an increment of force ΔW and we search for the next equilibrium point due to this new increment. We know that the solution lies on an extreme point (minimum or maximum).

Let us consider the known displacement vector $\vec{z}_0$ in n dimensions, which is represented by its components z_{0i} $(i=1,2,...,n)$. Let us suppose that $z_i = z_{0i} + \beta d_i$ where d_i represents a vector, supposedly known, that minimizes or maximizes the total potential energy. With that total potential energy is a function of β, i.e. $\Pi(z_i) = \overline{\Pi}(\beta)$.

We apply the chain rule in order to obtain:

$$\frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = \frac{\partial\overline{\Pi}(\beta)}{\partial z_i}\frac{\partial z_i}{\partial\beta} = \frac{\partial\overline{\Pi}(\beta)}{\partial z_i} d_i \tag{C.26}$$

or in tensorial notation:

$$\frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = \frac{\partial\overline{\Pi}(\beta)}{\partial\vec{z}} \cdot \frac{\partial\vec{z}}{\partial\beta} = \frac{\partial\overline{\Pi}(\beta)}{\partial\vec{z}} \cdot \vec{\boldsymbol{d}} = \nabla_{\vec{z}}\overline{\Pi}(\beta) \cdot \vec{\boldsymbol{d}} \tag{C.27}$$

where $\vec{\boldsymbol{g}} = \frac{\partial\overline{\Pi}(\beta)}{\partial\vec{z}} \equiv \nabla_{\vec{z}}\overline{\Pi}(\beta)$ represents the gradient of $\overline{\Pi}(\beta)$ in the space defined by $\vec{z}$.
We rewrite the above equation as follows:

$$\frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = \vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{d}} \xrightarrow{\textit{indicial}} \frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = g_i d_i \xrightarrow{\textit{matrix form}} \frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = \{\boldsymbol{g}\}^T\{\boldsymbol{d}\} \tag{C.28}$$

where $\vec{\boldsymbol{g}} = \vec{\boldsymbol{g}}(\vec{z})$ is the residue vector.

Taking once again the derivative of the equation (C.26) we can obtain:

$$\begin{aligned} \frac{\mathrm{d}^2\overline{\Pi}(\beta)}{\mathrm{d}\beta^2} &= \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{\partial\overline{\Pi}(\beta)}{\partial z_i} d_i\right) = \left(\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\partial\overline{\Pi}(\beta)}{\partial z_i} d_i + \frac{\partial\overline{\Pi}(\beta)}{\partial z_i}\underbrace{\frac{\mathrm{d}d_i}{\mathrm{d}\beta}}_{=0_i}\right) \\ &= \frac{\partial}{\partial z_i}\left(\frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta}\right) d_i = \frac{\partial}{\partial z_i}\left(\frac{\partial\overline{\Pi}(\beta)}{\partial z_j} d_j\right) d_i = d_i \frac{\partial^2\overline{\Pi}(\beta)}{\partial z_i \partial z_j} d_j = d_i S_{ij} d_j \end{aligned} \tag{C.29}$$

or in tensorial notation:

$$\frac{\mathrm{d}^2\overline{\Pi}(\beta)}{\mathrm{d}\beta^2} = \vec{\boldsymbol{d}} \cdot \frac{\partial^2\overline{\Pi}(\beta)}{\partial\vec{z} \otimes \partial\vec{z}} \cdot \vec{\boldsymbol{d}} = \vec{\boldsymbol{d}} \cdot \boldsymbol{S} \cdot \vec{\boldsymbol{d}} \xrightarrow{\textit{matrix form}} \frac{\mathrm{d}^2\overline{\Pi}(\beta)}{\mathrm{d}\beta^2} = \{\boldsymbol{d}\}^T[\boldsymbol{S}]\{\boldsymbol{d}\} \tag{C.30}$$

where the matrix $\boldsymbol{S}$ is the Hessian of $\overline{\Pi}(\beta)$, and within the scope of structural analysis is called tangent stiffness matrix.

Next, we will express $\overline{\Pi}(z_{0i} + \beta d_i)$ my means of Taylor series:

$$\overline{\Pi}(z_{0i} + \beta d_i) = \overline{\Pi}(\vec{z}_0) + \frac{\partial\overline{\Pi}(\vec{z}_0)}{\partial\beta}\beta + \frac{1}{2!}\frac{\partial^2\overline{\Pi}(\vec{z}_0)}{\partial\beta^2}\beta^2 + ... \tag{C.31}$$

and taking into account $\frac{\mathrm{d}\overline{\Pi}(\beta)}{\mathrm{d}\beta} = g_i d_i$ and $\frac{\mathrm{d}^2\overline{\Pi}(\beta)}{\mathrm{d}\beta^2} = d_i S_{ij} d_j$, the above equation becomes

$$\overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})=\overline{\Pi}(\vec{z}_0)+g_i d_i\beta+0.5 d_i S_{ij} d_j \beta^2$$
$$\Rightarrow \overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\Delta\overline{\Pi}=\beta(g_i d_i+0.5\beta d_i S_{ij} d_j) \tag{C.32}$$

$$\xrightarrow{tensorial} \quad =\Delta\overline{\Pi}=\beta(\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}+0.5\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}})$$
$$\xrightarrow{matrix\ form} \quad =\Delta\overline{\Pi}=\beta\left(\{\boldsymbol{g}\}^T\{\boldsymbol{d}\}+0.5\beta\{\boldsymbol{d}\}^T[\boldsymbol{S}]\{\boldsymbol{d}\}\right) \tag{C.33}$$

We can deal with two scenarios:

Minimum

$$\overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})<\overline{\Pi}(\vec{z}_0) \quad \Rightarrow \quad \overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\beta(g_i d_i+0.5\beta d_i S_{ij} d_j)<0$$
$$\Rightarrow \overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\beta(\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}+0.5\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}})=\beta(\vec{\boldsymbol{g}}+0.5\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S})\cdot\vec{\boldsymbol{d}}<0 \tag{C.34}$$

In this situation, $\boldsymbol{S}$ is positive definite, i.e. all its eigenvalues are positive, and as consequence $d_i S_{ij} d_j > 0$. with that we can conclude that $g_i d_i < 0$ and is the predominate term for small values of $\beta > 0$:

$$g_i d_i < 0 \quad \xrightarrow{tensorial} \quad \vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}<0 \tag{C.35}$$

Maximum

$$\overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})>\overline{\Pi}(\vec{z}_0) \quad \Rightarrow \quad \overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\beta(g_i d_i+0.5\beta d_i S_{ij} d_j)>0 \tag{C.36}$$

In this case, $\boldsymbol{S}$ is definite negative, i.e. all its eigenvalues are negative, and as consequence $d_i S_{ij} d_j < 0$. With that we can conclude that $g_i d_i > 0$ and it is the predominate term for small values of $\beta > 0$:

$$g_i d_i > 0 \quad \xrightarrow{tensorial} \quad \vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}>0 \tag{C.37}$$

For better illustration of the previous development, we will make an analogy where we have a mountain and depression in which they are represented by its level curves, (see Figure C.16). Recall that the gradient always points towards the growing sense of the function and is normal to the level curves.

If the parameters $\vec{\boldsymbol{g}}$ and $\vec{\boldsymbol{d}}$ are fixed, the value of β that minimize or maximize the function $\Delta\overline{\Pi}(\beta)$, (see equation (C.34)), is given by:

$$\overline{\Pi}(\vec{z}_0+\beta\vec{\boldsymbol{d}})-\overline{\Pi}(\vec{z}_0)=\Delta\overline{\Pi}=\beta(\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}+0.5\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}})$$
$$\Rightarrow \frac{\mathrm{d}\Delta\overline{\Pi}}{\mathrm{d}\beta}=\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}+\beta\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}}=0$$
$$\Rightarrow \beta=\frac{-\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}}{\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}}} \tag{C.38}$$

Given a known state $\vec{z}_0$, now the question is: Which is the value of $\vec{\boldsymbol{d}}$ in which we must adopt in order to guarantee that we are reaching an extreme?

If we are looking for a minimum, it is enough to adopt a matrix $\boldsymbol{A}$ which is positive definite, with that the equation $\vec{\boldsymbol{g}}\cdot\boldsymbol{A}\cdot\vec{\boldsymbol{g}}>0$ holds. Then, the vector $\vec{\boldsymbol{d}}$ could be defined such as:

$$\vec{\boldsymbol{d}}=-\boldsymbol{A}\cdot\vec{\boldsymbol{g}} \tag{C.39}$$

In this way we guarantee that:

$$\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}=-\vec{\boldsymbol{g}}\cdot\boldsymbol{A}\cdot\vec{\boldsymbol{g}}<0 \tag{C.40}$$

The choice for the matrix $\boldsymbol{A}$ is what given origin for several iterative methods.

Taking into account that given a matrix which is positive definite its inverse is also definite positive, we adopt $\boldsymbol{A}=\boldsymbol{S}^{-1}$, with that $\vec{\boldsymbol{d}}=-\boldsymbol{A}\cdot\vec{\boldsymbol{g}}=-\boldsymbol{S}^{-1}\cdot\vec{\boldsymbol{g}}\Rightarrow\vec{\boldsymbol{g}}=-\boldsymbol{S}\cdot\vec{\boldsymbol{d}}$. Then, we can conclude that:

$$\beta=\frac{-\vec{\boldsymbol{g}}\cdot\vec{\boldsymbol{d}}}{\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}}}=\frac{-(-\boldsymbol{S}\cdot\vec{\boldsymbol{d}})\cdot\vec{\boldsymbol{d}}}{\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}}}=\frac{\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}}}{\vec{\boldsymbol{d}}\cdot\boldsymbol{S}\cdot\vec{\boldsymbol{d}}}=1 \tag{C.41}$$

With these conditions we obtain the Newton-Raphson's method, in which the matrix $\boldsymbol{A}$ is the inverse of the Hessian matrix (tangent stiffness matrix) $\boldsymbol{A}=\boldsymbol{S}^{-1}$.

$$\beta^{(k)}=\frac{-\vec{\boldsymbol{g}}^{(k)}\cdot\vec{\boldsymbol{d}}^{(k)}}{\vec{\boldsymbol{d}}^{(k)}\cdot\boldsymbol{S}^{(k)}\cdot\vec{\boldsymbol{d}}^{(k)}}=\frac{-\vec{\boldsymbol{g}}^{(k)}\cdot\vec{\boldsymbol{d}}^{(k)}}{\vec{\boldsymbol{d}}^{(k)}\cdot(\vec{\boldsymbol{g}}^{(k+1)}-\vec{\boldsymbol{g}}^{(k)})} \tag{C.42}$$

Note that $\boldsymbol{S}^{(k)}\cdot\vec{\boldsymbol{d}}^{(k)}=\vec{\boldsymbol{g}}^{(k+1)}-\vec{\boldsymbol{g}}^{(k)}$.

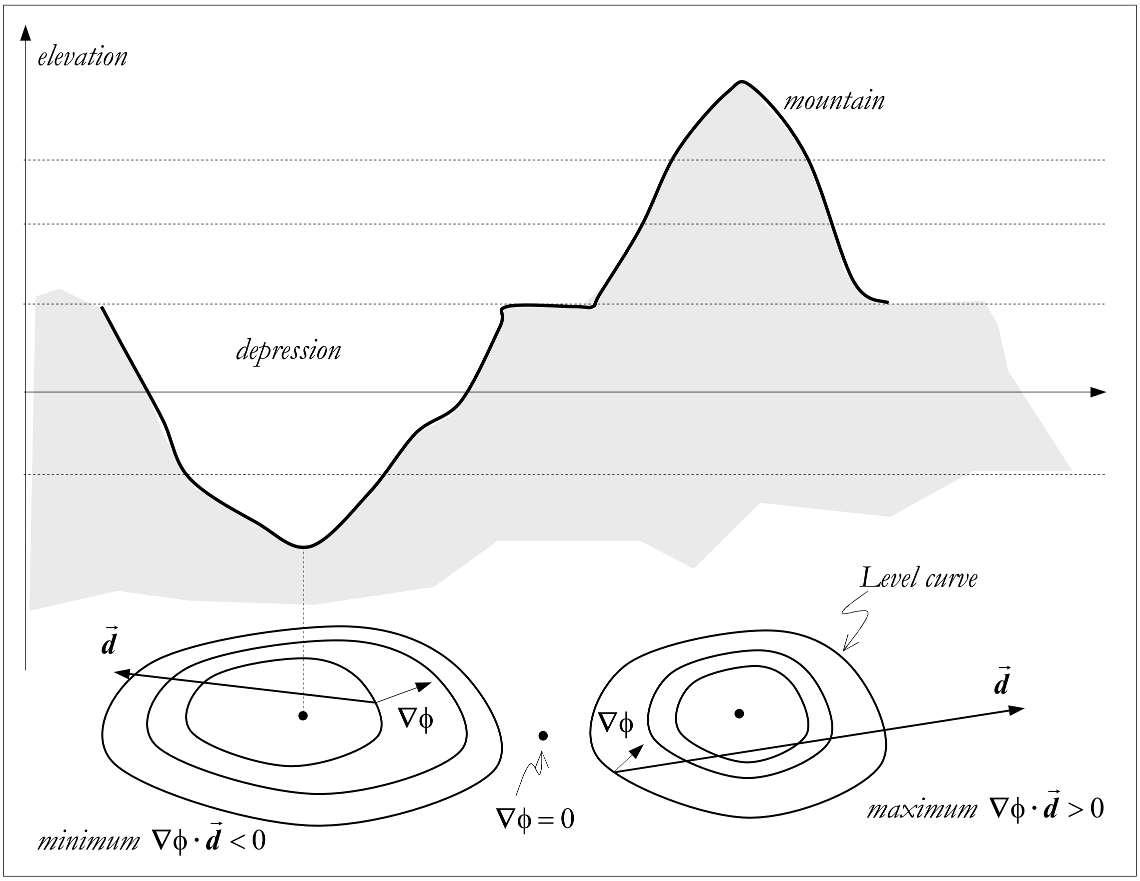

Figure C.16: Level curves.

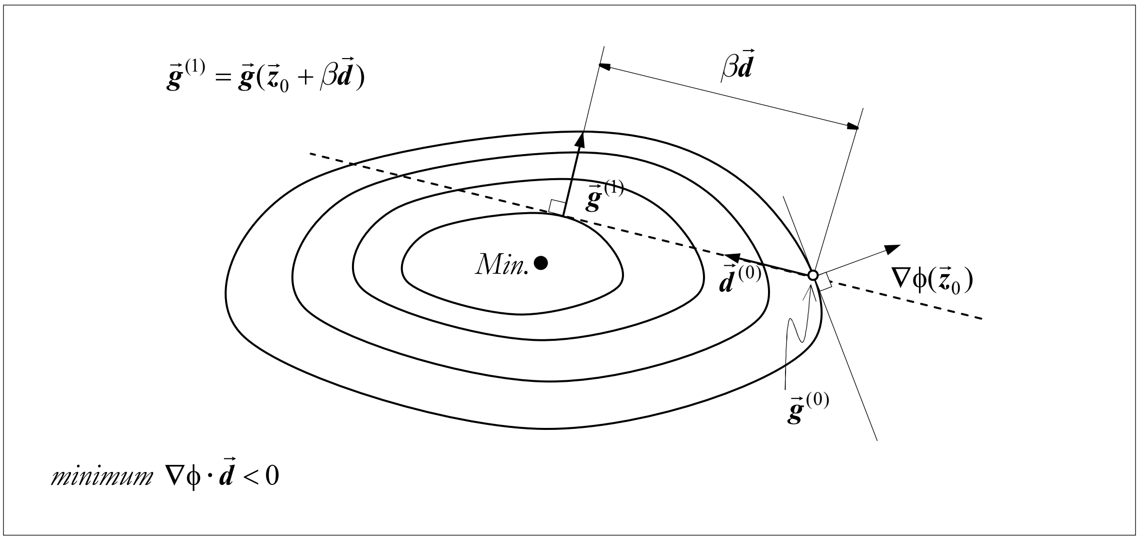

Figure C.17

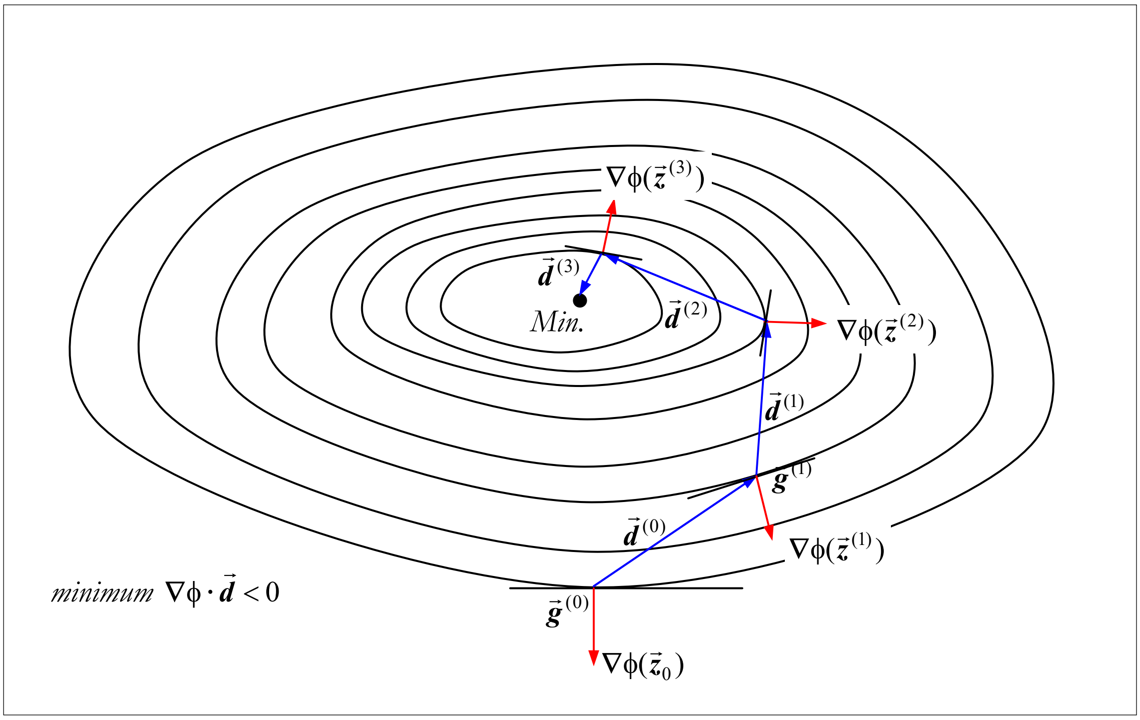

Figure C.18: Iterative process.

C.5 Classical Formulation of the Newton-Raphson's Method

The Newton-Raphson's is employed to obtain the roots of a function, i.e. $f(x)=0$, (Chapra&Canale(1988)). Newton in 1669 has obtained a version of the method and Raphson in 1690 has generalized the method.

The formulation of the Newton-Raphson's method can be obtained by means of Taylor series. For example, given the function $f(x)$ we can approach the value of the function $f(x+\Delta x)$ by Taylor series as follows:

$$
\begin{aligned}
f(x+\Delta x) &= f(x_0) + \frac{1}{1!}\frac{\partial f(x_0)}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 f(x_0)}{\partial x^2}\Delta x^2 + \dots \\
&\approx f(x_0) + f'(x_0)\Delta x
\end{aligned}
\tag{C.43}
$$

where we have considered until linear terms. Let us adopt the following nomenclature: $x_{i+1} = x_i + \Delta x$, $\Delta x = x_{i+1} - x_i$ and for the application point $x_i = x_0$. With that the above equation can be rewritten as follows:

$$
f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x = f(x_i) + f'(x_i)(x_{i+1} - x_i) \tag{C.44}
$$

Here the index (i) does not indicate indicial notation.

As we are searching for the roots of the function $f(x_{i+1}) = 0$ we can obtain:

$$
\begin{aligned}
& f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) = 0 \\
& \Rightarrow f(x_i) + x_{i+1} f'(x_i) - x_i f'(x_i) = 0 \\
& \Rightarrow x_{i+1} f'(x_i) = x_i f'(x_i) - f(x_i) \\
& \Rightarrow x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\end{aligned}
\tag{C.45}
$$

Once x_{i+1} is obtained the values of the function $(f(x_{i+1}))$ and its derivative $(f'(x_{i+1}))$ at the point can be obtained. This procedure is repeated until $f(x_{i+n}) \approx 0$ is reached. It can be shown that the error associated with it is proportionally to the square of the previous error, i.e. it has *quadratic convergence*. For more details the lector is referred to Chapra&Canale (1988).

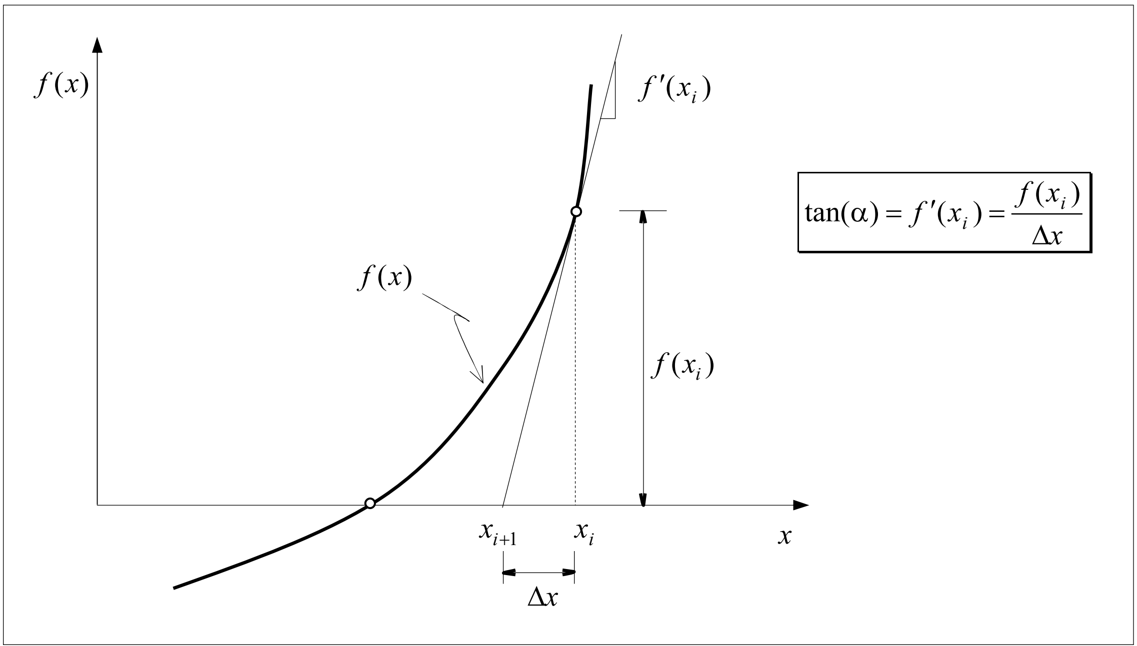

Figure C.19: Newton-Raphson's method.

In general, in structure analysis it is taken as a direct solution procedure to solve the set of equations. For a non-linear problem an incremental/iterative strategy is adopted. In this way we apply and increment of load (*load step*) and in each load step we apply iterative procedure in order to achieve the equilibrium point.

Let us consider that the function is the residue $\vec{\boldsymbol{g}}(\vec{\boldsymbol{u}})$ and we know its value and its derivative, $\frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}}$, at the application point $\vec{\boldsymbol{u}}_0$. Then, by means of Taylor series we can obtain:

$$\vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}) = \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0) + \frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}} \cdot \Delta \vec{\boldsymbol{u}} \tag{C.46}$$

or in indicial notation:

$$\begin{aligned} g_i(\vec{\boldsymbol{u}}) &= g_i(\vec{\boldsymbol{u}}_0) + \frac{\partial g_i(\vec{\boldsymbol{u}}_0)}{\partial u_j} \Delta u_j = g_i(\vec{\boldsymbol{u}}_0) + \frac{\partial g_i(\vec{\boldsymbol{u}}_0)}{\partial u_j}(u_j - u_{0j}) \\ &= g_i(\vec{\boldsymbol{u}}_0) - K_{ij}^{tan}(u_j - u_{0j}) \end{aligned} \tag{C.47}$$

where $\frac{\partial g_i(\vec{\boldsymbol{u}}_0)}{\partial u_j} = -K_{ij}^{tan}(\vec{\boldsymbol{u}}_0)$ is the Jacobian matrix which in structural analysis ambit represents the tangent stiffness matrix at the application point $\vec{\boldsymbol{u}}_0$. As we are looking for the value in which $\vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}) = \vec{\boldsymbol{0}}$, we can state that:

$$\begin{aligned} & g_i(\vec{\boldsymbol{u}}) = g_i(\vec{\boldsymbol{u}}_0) - K_{ij}^{tan}(u_j - u_{0j}) = 0_i \qquad \Rightarrow \qquad -g_i(\vec{\boldsymbol{u}}_0) = -K_{ij}^{tan}(u_j - u_{0j}) \\ & \Rightarrow -(K_{ki}^{tan})^{-1} g_i(\vec{\boldsymbol{u}}_0) = -(K_{ki}^{tan})^{-1} K_{ij}^{tan}(u_j - u_{0j}) \\ & \Rightarrow (K_{ki}^{tan})^{-1} g_i(\vec{\boldsymbol{u}}_0) = \delta_{kj}(u_j - u_{0j}) = (u_k - u_{0k}) \\ & \Rightarrow u_k = u_{0k} + (K_{ki}^{tan})^{-1} g_i(\vec{\boldsymbol{u}}_0) \end{aligned} \tag{C.48}$$

With that we can conclude:

$$u_i = u_{0i} + (K_{ij}^{tan})^{-1} g_j(\vec{\boldsymbol{u}}_0) = u_{0i} + \Delta u_i \tag{C.49}$$

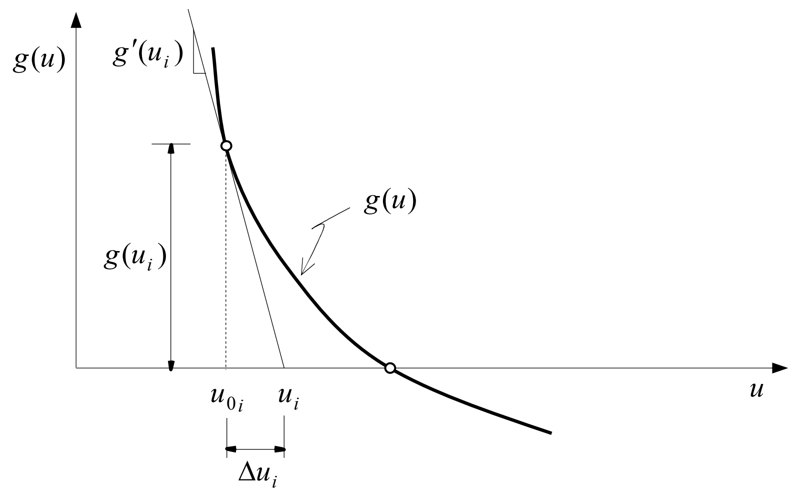

Figure C.20: Residue function.

We can generalized the equation in (C.46) such as:

$$\begin{aligned}
&\vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}) = \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0) + \frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}} \cdot \Delta \vec{\boldsymbol{u}} \\
&\Rightarrow \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}) - \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0) = \left(\frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}} \right) \cdot \Delta \vec{\boldsymbol{u}} \\
&\Rightarrow \Delta \vec{\boldsymbol{g}}^{(k)} = \left(\frac{\partial \vec{\boldsymbol{g}}(\vec{\boldsymbol{u}}_0)}{\partial \vec{\boldsymbol{u}}} \right)^{(k)} \cdot \Delta \vec{\boldsymbol{u}}^{(k)} \\
&\Rightarrow \Delta \vec{\boldsymbol{g}}^{(k)} = \boldsymbol{H}^{(k)} \cdot \Delta \vec{\boldsymbol{u}}^{(k)}
\end{aligned} \tag{C.50}$$

where (k) means iterations. When $\boldsymbol{H}^{(k)}$ represents the tangent matrix and changes for each iteration we fallback into the classical Newton-Raphson's method. When the matrix $\boldsymbol{H}^{(k)}$ does not change during the iterations we are dealing with the so-called Modified Newton-Raphson's method. We can also state that:

$$\begin{aligned}
&\Delta \vec{\boldsymbol{g}}^{(k)} = \boldsymbol{H}^{(k)} \cdot \Delta \vec{\boldsymbol{u}}^{(k)} \\
&\Rightarrow \Delta \vec{\boldsymbol{g}}^{(k)} = (\boldsymbol{H}^{(k-1)} + \boldsymbol{E}^{(k)}) \cdot \Delta \vec{\boldsymbol{u}}^{(k)}
\end{aligned} \tag{C.51}$$

where $\boldsymbol{E}^{(k)}$ is a correction matrix. In a generic way we can adopt:

$$\boldsymbol{E}^{(k)} = \alpha \vec{\boldsymbol{a}} \otimes \vec{\boldsymbol{a}} + \beta \vec{\boldsymbol{b}} \otimes \vec{\boldsymbol{b}} \tag{C.52}$$

where we can adopt $\vec{\boldsymbol{a}} = \vec{\boldsymbol{u}}^{(k)}$, $\vec{\boldsymbol{b}} = \boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}$. With that the above equation becomes:

$$\boldsymbol{E}^{(k)} = \alpha \vec{\boldsymbol{u}}^{(k)} \otimes \vec{\boldsymbol{u}}^{(k)} + \beta (\boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}) \otimes (\boldsymbol{H}^{(k-1)} \cdot \vec{\boldsymbol{g}}^{(k)}) \tag{C.53}$$

The values for α and β can be determined in a such a way that the equation in (C.51) is fulfilled. For example:

$$\alpha \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{g}}^{(k)} = 1 \quad \Rightarrow \quad \alpha = \frac{1}{\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{g}}^{(k)}} = \frac{1}{\vec{\boldsymbol{u}}^{(k)} \cdot \vec{\boldsymbol{g}}^{(k)}} \tag{C.54}$$

$$\beta\vec{b}\cdot\vec{g}^{(k)} = -1 \quad \Rightarrow \quad \beta = \frac{-1}{\vec{b}\cdot\vec{g}^{(k)}} = \frac{-1}{(\boldsymbol{H}^{(k-1)}\cdot\vec{g}^{(k)})\cdot\vec{g}^{(k)}} = \frac{-1}{\vec{g}^{(k)}\cdot\boldsymbol{H}^{(k-1)}\cdot\vec{g}^{(k)}} \tag{C.55}$$

In this way we can obtain:

$$\boldsymbol{E}^{(k)} = \frac{\vec{u}^{(k)}\otimes\vec{u}^{(k)}}{\vec{u}^{(k)}\cdot\vec{g}^{(k)}} - \frac{(\boldsymbol{H}^{(k-1)}\cdot\vec{g}^{(k)})\otimes(\boldsymbol{H}^{(k-1)}\cdot\vec{g}^{(k)})}{\vec{g}^{(k)}\cdot\boldsymbol{H}^{(k-1)}\cdot\vec{g}^{(k)}} \tag{C.56}$$

And in turn we define the matrix $\boldsymbol{H}^{(k)}$ as follows:

$$\begin{aligned} &\boldsymbol{H}^{(k)} = \boldsymbol{H}^{(k-1)} + \boldsymbol{E}^{(k)} \\ &\Rightarrow \boldsymbol{H}^{(k)} = \boldsymbol{H}^{(k-1)} + \frac{\vec{u}^{(k)}\otimes\vec{u}^{(k)}}{\vec{u}^{(k)}\cdot\vec{g}^{(k)}} - \frac{(\boldsymbol{H}^{(k-1)}\cdot\vec{g}^{(k)})\otimes(\boldsymbol{H}^{(k-1)}\cdot\vec{g}^{(k)})}{\vec{g}^{(k)}\cdot\boldsymbol{H}^{(k-1)}\cdot\vec{g}^{(k)}} \end{aligned} \tag{C.57}$$

This method is called BFGS method (Broyden-Fletcher-Goldfarb-Shanno). Note that if we know the inverse of $\boldsymbol{H}^{(k-1)}$, the inverse of $\boldsymbol{H}^{(k)}$ can be easily obtained by means of **Problem 1.117**.

C.6 Newton-Raphson's Method

Let us return to our initial problem proposed in Section C.2 and let us assume that we know the values W_n and z_n. For the force increment ΔW_{n+1} we can state that $W_{n+1} = W_n + \Delta W_{n+1}$. For the next load step, represented by $n+1$, we have that for the first iteration (i):

PREDICTION $(i=0)$

Tangent matrix calculation:

$$N^i_{n+1} = EA\left[\left(\frac{z_0}{L_0}\right)\left(\frac{z^i_{n+1}}{L_0}\right) + \frac{1}{2}\left(\frac{z^i_{n+1}}{L_0}\right)^2\right] \xrightarrow{if \quad i=1} N^i_{n+1} \leftarrow N_n = EA\left[\left(\frac{z_0}{L_0}\right)\left(\frac{z_n}{L_0}\right) + \frac{1}{2}\left(\frac{z_n}{L_0}\right)^2\right] \tag{C.58}$$

$$\begin{aligned} &(K^{tan})^i_{n+1} = \frac{EA}{L_0^3}(z_0 + z^i_{n+1})^2 + \frac{N^i_{n+1}}{L_0} + K_S \\ &if \quad i=1 \Rightarrow (K^{tan})^i_{n+1} \leftarrow (K^{tan})_n = \frac{EA}{L_0^3}(z_0 + z_n)^2 + \frac{N_n}{L_0} + K_S \end{aligned} \tag{C.59}$$

Solve the system:

$$\Delta z^i_{n+1} = \left(K^{tan\,i}_{n+1}\right)^{-1}\mathcal{R}^i_{n+1} \xrightarrow{if \quad i=1} \Delta z^i_{n+1} = \left(K^{tan}_n\right)^{-1}\Delta W_{n+1} \tag{C.60}$$

where $\mathcal{R}^i_{n+1}$ is the residue. Once the above equation is solved, we can obtain:

$$z^i_{n+1} = z_n + \Delta z^i_{n+1} \tag{C.61}$$

And in turn we can calculate the internal forces for this iteration:

$$N_{n+1}^{i} = EA\left[\left(\frac{z_0}{L_0}\right)\left(\frac{z_{n+1}^{i}}{L_0}\right) + \frac{1}{2}\left(\frac{z_{n+1}^{i}}{L_0}\right)^2\right] \tag{C.62}$$

and

$$\overline{W}_{n+1}^{i} = N_{n+1}^{i}\frac{(z_0 + z_{n+1}^{i})}{L_0} + K_S z_{n+1}^{i} \tag{C.63}$$

and the residue can be obtained:

$$\mathcal{R}_{n+1}^{i=2} = \overline{W}_{n+1}^{i=1} - W_{n+1} \tag{C.64}$$

Next, we obtain a norm in order to check the convergence. For our one-dimensional example the residue is a scalar, and we check whether this norm is less than a tolerance. If so, we go to the next load step $(n \leftarrow n+1)$. Otherwise, we have to do a new iteration $(i \leftarrow i+1)$ until the convergence is achieved, (see Figure C.21).

The procedure in FORTRAN code can be found at the link: NON_LINEAR1D.FOR, (starting at *label 40).*

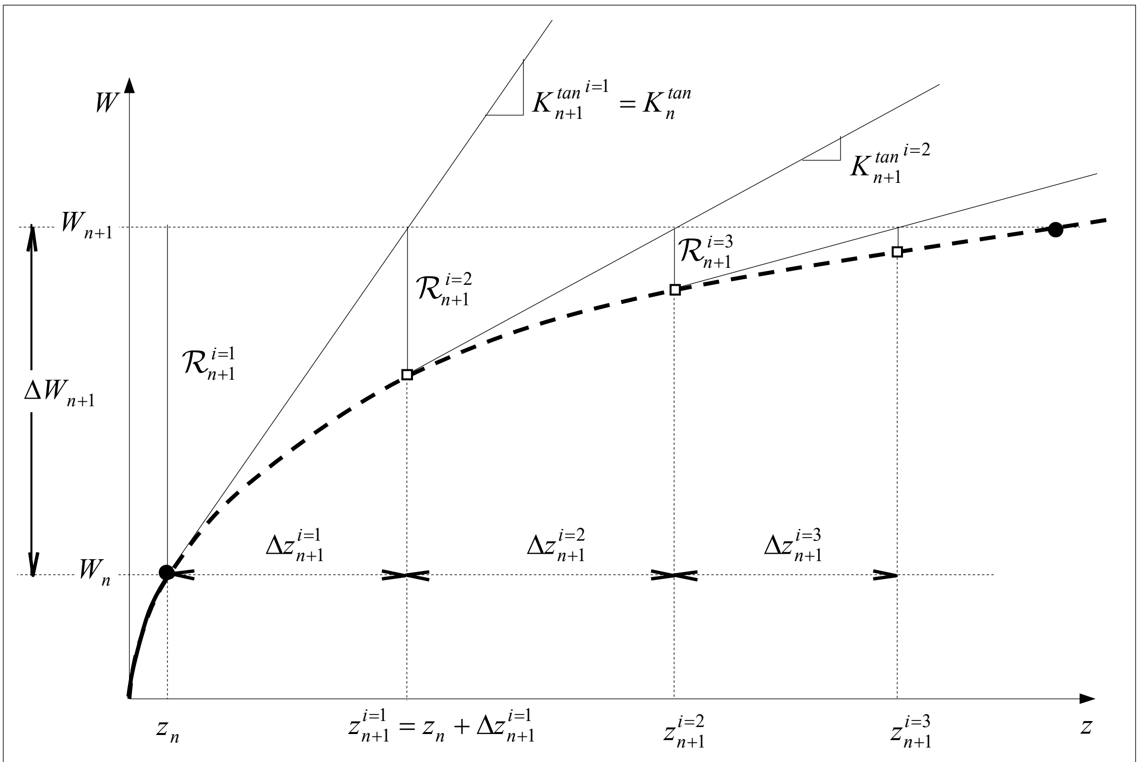

Figure C.21: Newton-Raphson iterative procedure.

For our example we will consider an load increment equal to $\Delta W = 30$. With that we can obtain the graph describe in Figure C.22. And in Figure C.23 the curve for several load steps.

As we can see in Figure C.22, for each iteration, we need to calculate the tangent matrix. This is a drawback of the Newton-Raphson's method, since for a system with a large number of variables it could be costly, from a computation point of view, to calculate the stiffness matrix for each iteration.

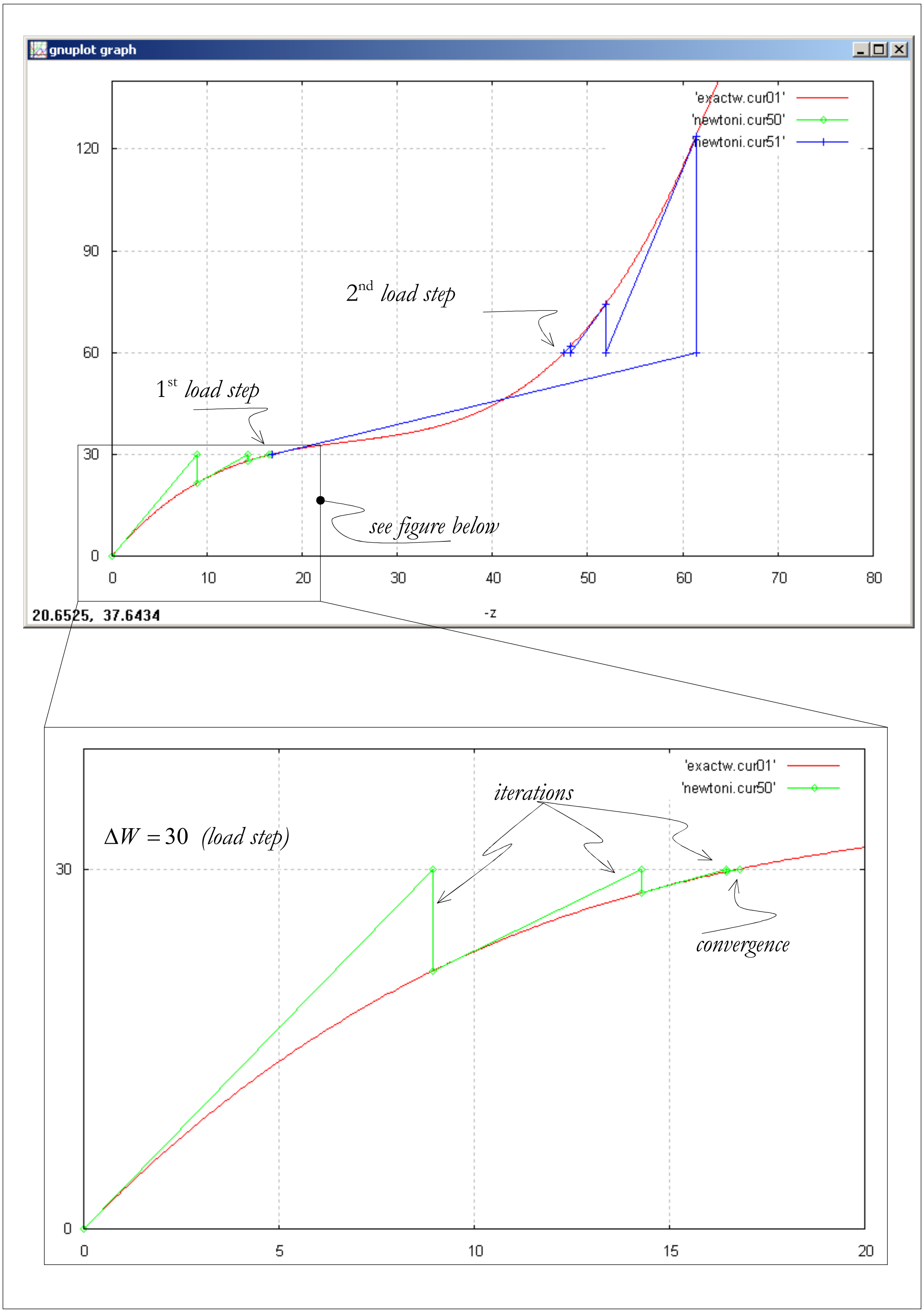

Figure C.22: Newton-Raphson iterative procedure.

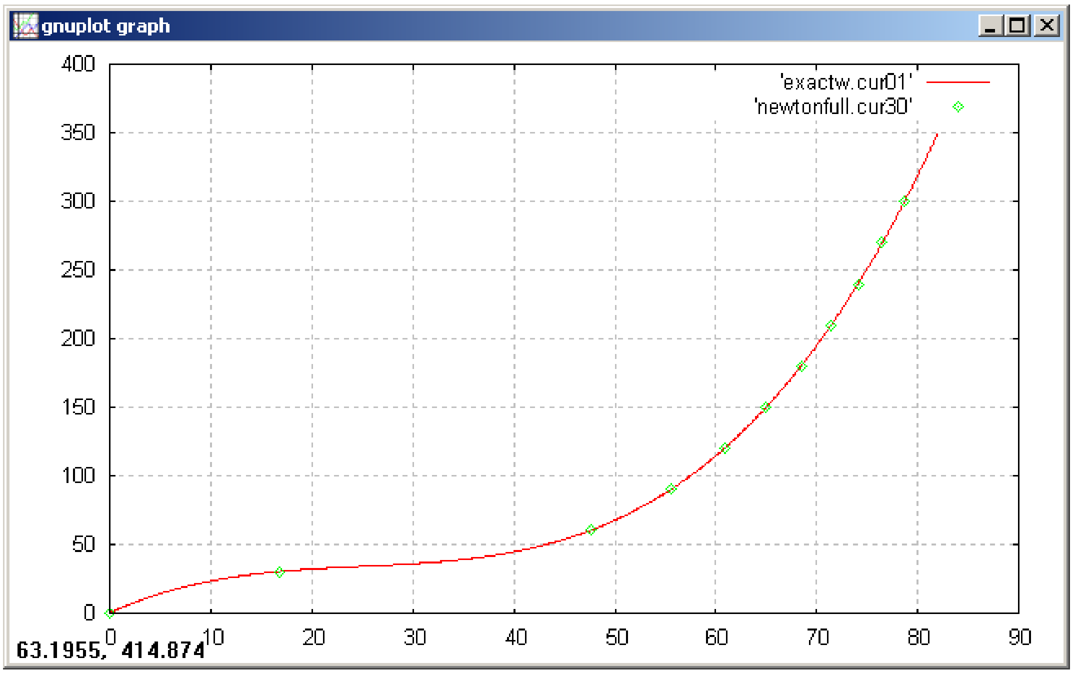

Figure C.23: Example of Newton-Raphson iterative procedure.

C.7 Modified Newton-Raphson Method

Several methods have been formulated based on the Newton-Raphson's method, e.g. the modified Newton-Raphson method, which basically consists in adopt the same tangent matrix for each iteration, (see Figure C.24). This method requires more iterations than the Full Newton-Raphson method and beside has no convergence if we are dealing with inflection point as the one describe in Figure C.22. Just to illustrate this method, we apply the load increment equal to $\Delta W = 30$ which is before the inflection point, (see Figure C.25). As we can see the modified Newton-Raphson method needs more iterations to achieve convergence.

The procedure in FORTRAN code can be found at the link: NON_LINEAR1D.FOR, (starting at *label 50)*.

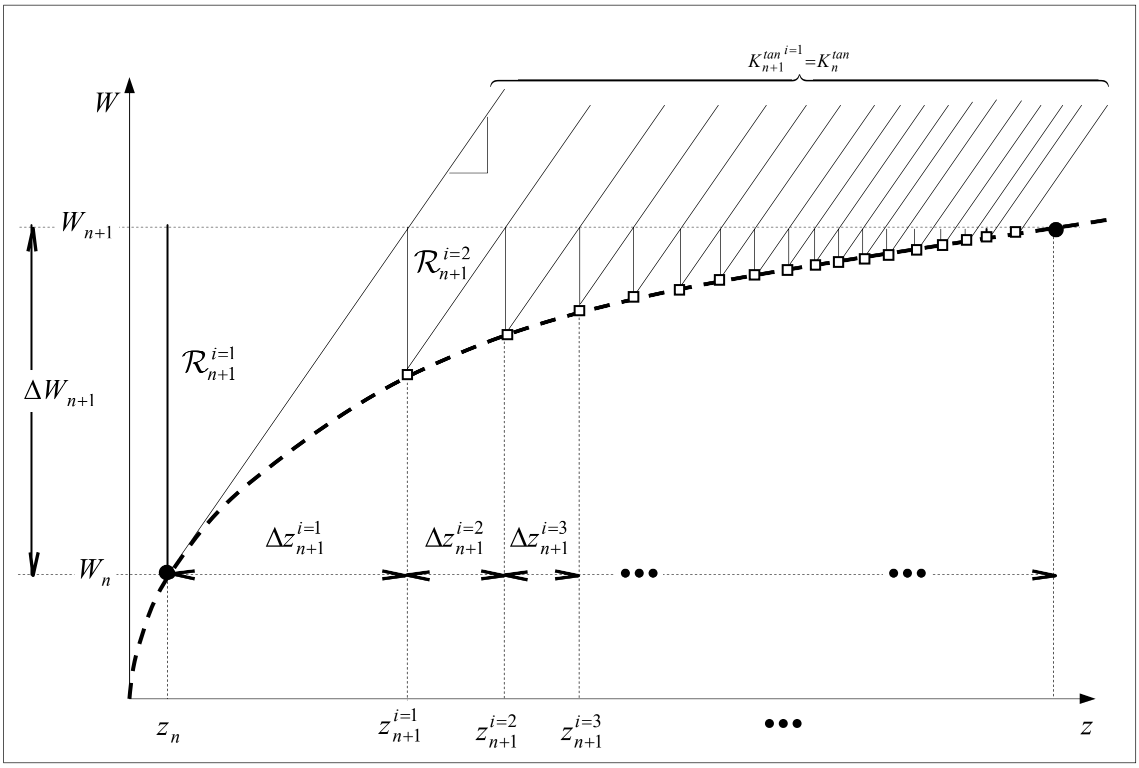

Figure C.24: Modified Newton-Raphson iterative procedure.

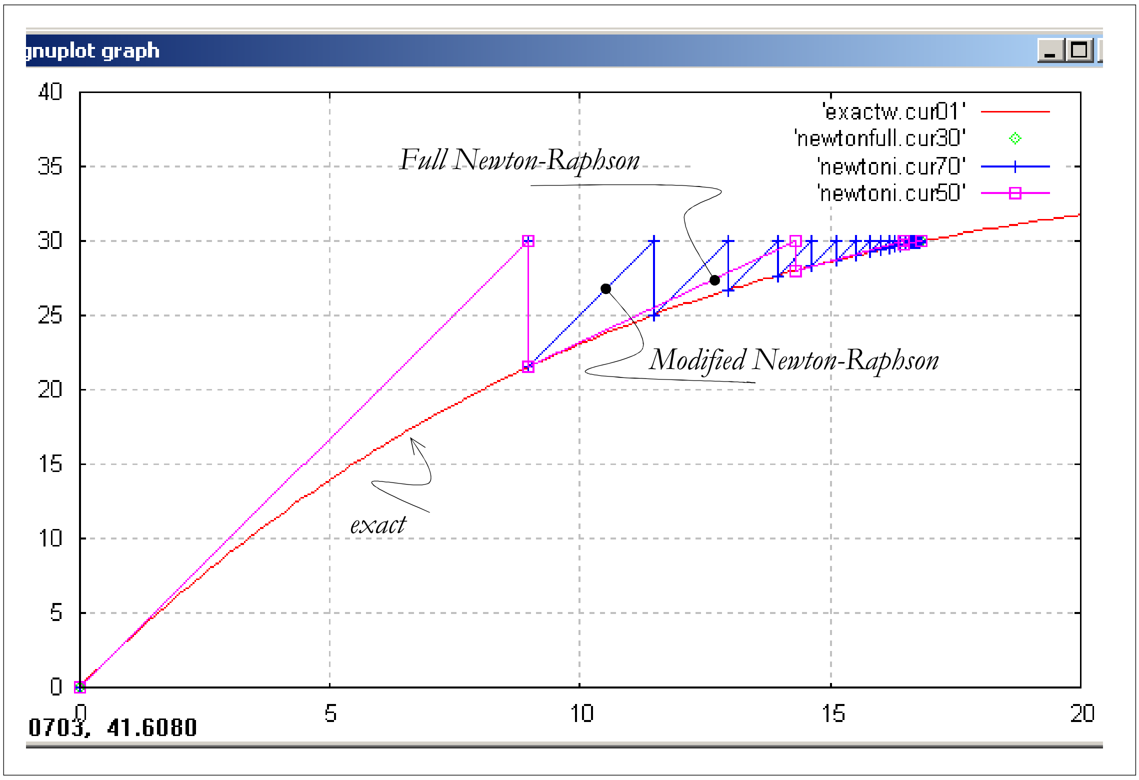

Figure C.25: Full Newton-Raphson *vs.* Modified Newton-Raphson.

Incremental-Iterative Solution References

CHAPRA, S.C. & CANALE, R.P. (1988). *Métodos Numéricos para Ingenieros. Com aplicaciones en computadores personales*. McGraw-Hill/Interamericana de México, S.A. de C.V.

GANABA, T.H. (1985). *Nonlinear Finite Element Analysis of Plates and Slabs*. Ph.D. Thesis. The University of Warwick, Coventry, UK.

POSADA, L.M. (2007). *Stability Analysis of Two-dimensional Truss Structures*. MSc, Universität Stuttgart.

CRISFIELD, M.A.(1991). *Non-linear finite element analysis of solids and structures*. Vol.1. John Wiley & Sons, England.

CRISFIELD, M.A.(1997). *Non-linear finite element analysis of solids and structures*. Vol.2. John Wiley & Sons, England.

RUBERT, J.B. (1993). *Estudo do desempenho de algoritmos numéricos na solução de sistemas não-lenares de estruturas formadas por barras de treliça*. Dissertação de Mestrado, USP-Sao Carlos