Annex B

Introduction to Finite Differences

B.1 Introduction

The finite difference method was the first technique that emerged to solve numerically partial differential equations related to practical engineering problems. Today this technique is now obsolete with respect to solving partial differential equations, for instance, to solve problems related to beams, plates, flux, etc. But the finite difference technique is widely spread when we are dealing with numerical integration over time, (see Annex A).

B.2 The Finite Difference Method

Let us consider the function $y = y(x)$, and the derivative of y with respect to x is defined by:

$$y' \equiv \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x+\Delta x) - y(x)}{\Delta x} \tag{B.1}$$

where y' indicates the slope of the function at the point x, (see Figure B.1).

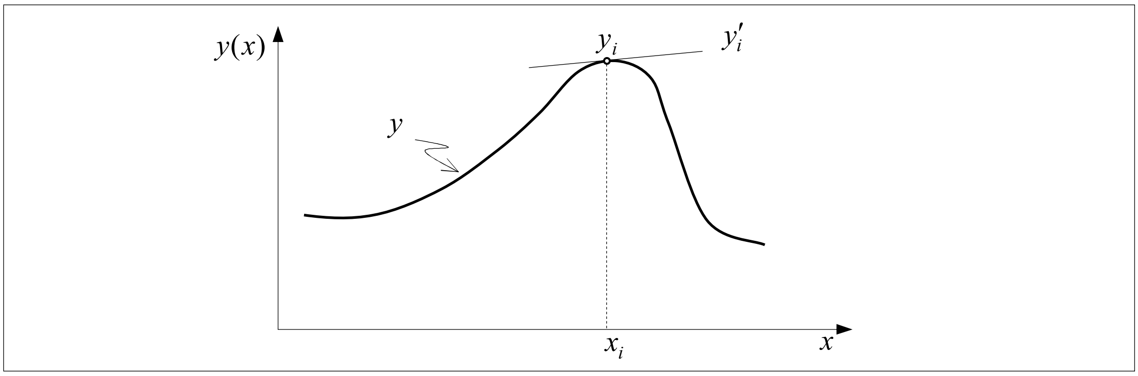

Figure B.1: Derivative of a function.

When the term Δx does not tend to zero but to a finite value, (see Figure B.2), the derivative at the point x_i can be defined in several ways. If we use the left neighbor point (y_{i-1}), *left finite difference*, the first derivative can be approached as follows:

$$y_i'^L = \left(\frac{\Delta y}{\Delta x}\right)_i = \frac{y_i - y_{i-1}}{\Delta x} \tag{B.2}$$

Otherwise, if we used information of the right neighbor point (y_{i+1}), *right finite difference*, the first derivative can be represented as follows:

$$y_i'^R = \left(\frac{\Delta y}{\Delta x}\right)_i = \frac{y_{i+1} - y_i}{\Delta x} \tag{B.3}$$

where we have adopted the nomenclature $y(x_{i-1}) = y_{i-1}$, $y(x_i) = y_i$, $y(x_{i+1}) = y_{i+1}$. As we can appreciate in Figure B.2, by using these techniques the first derivate is approximated to its exact value, i.e. there is an error associated with it. When $\Delta x \to 0$, the exact value for the first derivative is achieved.

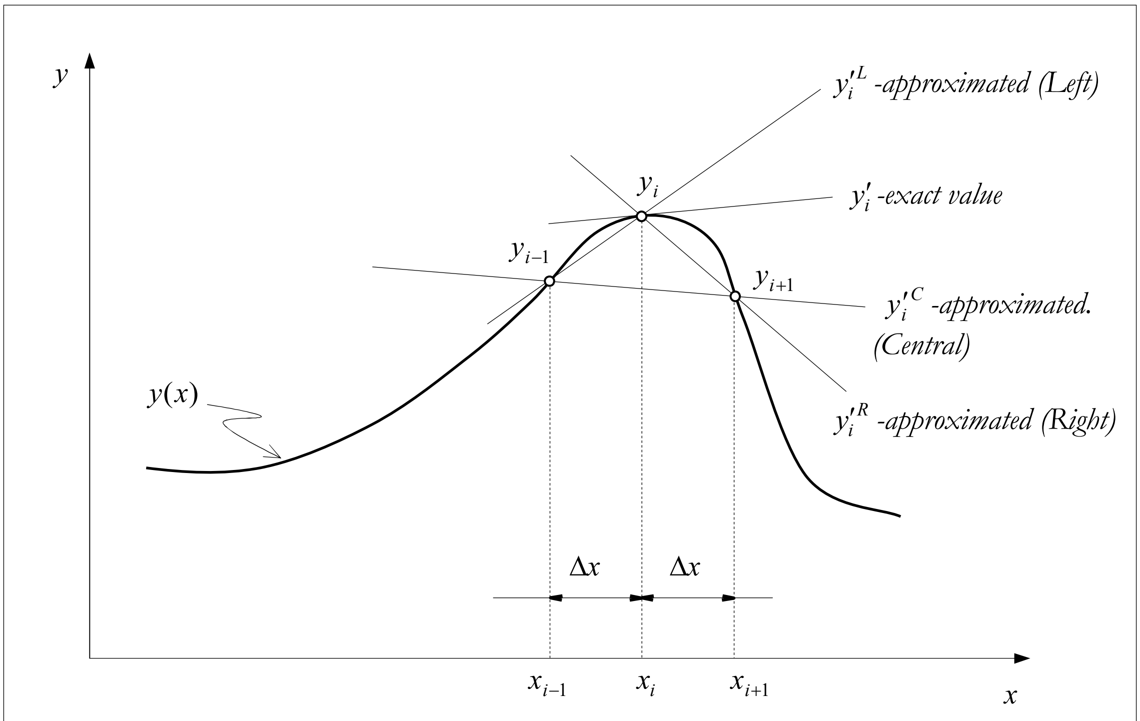

Figure B.2: Derivative of a function by means of finite differences.

We can still raise another possibility to approach the first derivative of the function at the point x_i by using simultaneously the left and the right points, *(central finite difference)*:

$$y_i'^C = \left(\frac{\Delta y}{\Delta x}\right)_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x} \tag{B.4}$$

As we can appreciate in Figure B.2, the central finite difference approach is closer to the exact value. Note that the central finite difference, for the first derivative, is the average of left and right techniques:

$$\left(\frac{\Delta y}{\Delta x}\right)_i = \frac{y_i'^R + y_i'^L}{2} = \frac{y_{i+1} - y_{i-1}}{2\Delta x} \tag{B.5}$$

Similarly we can obtain the derivatives for higher order, for example for the second derivative:

$$\frac{d^2y}{dx^2} = \lim_{\Delta x\to 0}\frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x\to 0}\frac{\frac{y(x+\Delta x)-y(x)}{\Delta x} - \frac{y(x)-y(x+\Delta x)}{\Delta x}}{\Delta x} \tag{B.6}$$

The left finite derivative:

$$\begin{aligned}\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i &= \frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right)^L = \frac{\Delta}{\Delta x}\left(\frac{y_i - y_{i-1}}{\Delta x}\right) = \frac{1}{\Delta x}\left(\frac{\Delta y_i}{\Delta x} - \frac{\Delta y_{i-1}}{\Delta x}\right)\\ &= \frac{1}{\Delta x}\left(\frac{y_i - y_{i-1}}{\Delta x} - \frac{y_{i-1}-y_{i-2}}{\Delta x}\right)\\ &= \frac{y_i - 2y_{i-1} + y_{i-2}}{\Delta x^2}\end{aligned} \tag{B.7}$$

The right finite derivative:

$$\begin{aligned}\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i &= \frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right)^R = \frac{\Delta}{\Delta x}\left(\frac{y_{i+1} - y_i}{\Delta x}\right) = \frac{1}{\Delta x}\left(\frac{\Delta y_{i+1}}{\Delta x} - \frac{\Delta y_i}{\Delta x}\right)\\ &= \frac{1}{\Delta x}\left(\frac{y_{i+2} - y_{i+1}}{\Delta x} - \frac{y_{i+1}-y_i}{\Delta x}\right) = \frac{y_{i+2} - 2y_{i+1} + y_i}{\Delta x^2}\end{aligned} \tag{B.8}$$

And by means of the central finite difference we can approach the second derivative as follows:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\frac{y_{i+1}-y_i}{\Delta x} - \frac{y_i - y_{i-1}}{\Delta x}}{\Delta x} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2} \tag{B.9}$$

B.2.1 Left Finite Differences

Next we will define an automatic way in order to obtain the operators $\Delta y, \Delta^2 y, \cdots$ when we use the left finite difference. As we have seen previously, for the first derivative we have $\Delta y = y_i - y_{i-1}$, (see equation (B.2)). If we want to obtain the operator for the second derivative we use the points located at the left side of the point x_i:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right) = \frac{\Delta}{\Delta x}\left(\frac{y_i - y_{i-1}}{\Delta x}\right) = \frac{\Delta y_i - \Delta y_{i-1}}{\Delta x^2} \tag{B.10}$$

By applying once more the left derivative definition we get $\Delta y_i = y_i - y_{i-1}$ and $\Delta y_{i-1} = y_{i-1} - y_{i-2}$ and by substituting into the above equation we can obtain:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\Delta y_i - \Delta y_{i-1}}{\Delta x^2} = \frac{(y_i - y_{i-1}) - (y_{i-1} - y_{i-2})}{\Delta x^2} = \frac{(y_i - 2y_{i-1} + y_{i-2})}{\Delta x^2} \tag{B.11}$$

Then, we define the operator $\Delta^2 y = y_i - 2y_{i-1} + y_{i-2}$ for the left finite difference case. In Figure B.3 we constructed a figure in order to obtain automatically the operators for higher order operators.

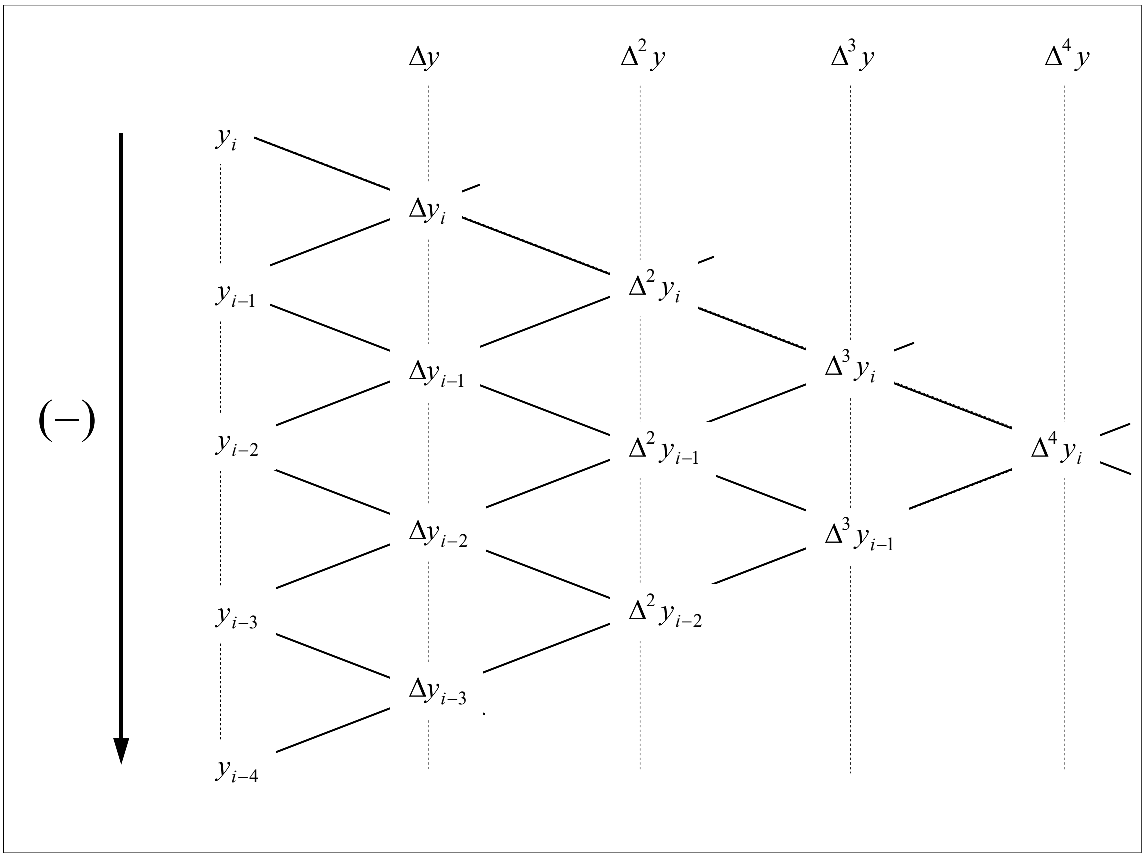

Figure B.3: Left finite differences.

For example, in order to obtain the operator $\Delta^4 y$ by means of Figure B.3 we localize the term $\Delta^4 y_i$ and we subtract the values as follows:

$$
\begin{aligned}
\Delta^4 y &= \Delta^3 y_i - \Delta^3 y_{i-1} = \left(\Delta^2 y_i - \Delta^2 y_{i-1}\right) - \left(\Delta^2 y_{i-1} - \Delta^2 y_{i-2}\right) = \Delta^2 y_i - 2\Delta^2 y_{i-1} + \Delta^2 y_{i-2} \\
&= \left(\Delta y_i - \Delta y_{i-1}\right) - 2\left(\Delta y_{i-1} - \Delta y_{i-2}\right) + \left(\Delta y_{i-2} - \Delta y_{i-3}\right) \\
&= \Delta y_i - 3\Delta y_{i-1} + 3\Delta y_{i-2} - \Delta y_{i-3} \\
&= \left(y_i - y_{i-1}\right) - 3\left(y_{i-1} - y_{i-2}\right) + 3\left(y_{i-2} - y_{i-3}\right) - \left(y_{i-3} - y_{i-4}\right) \\
&= y_i - 4y_{i-1} + 6y_{i-2} - 4y_{i-3} + y_{i-4}
\end{aligned}
\tag{B.12}
$$

With that we can define the fourth derivative by means of left finite difference as follows:

$$
\left(\frac{\Delta^4 y}{\Delta x^4}\right)_i = \frac{y_i - 4y_{i-1} + 6y_{i-2} - 4y_{i-3} + y_{i-4}}{\Delta x^4}
\tag{B.13}
$$

B.2.2 Right Finite Differences

Next we will define an automatic way in order to obtain the operators $\Delta y, \Delta^2 y, \cdots$ when we use the right finite difference. As we have seen previously, for the first derivative we have $\Delta y = y_{i+1} - y_i$, (see equation (B.3)). If we want to obtain the operator for the second derivative we use the points located at the right side of the point x_i:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\Delta}{\Delta x}\left(\frac{\Delta y}{\Delta x}\right) = \frac{\Delta}{\Delta x}\left(\frac{y_{i+1} - y_i}{\Delta x}\right) = \frac{\Delta y_{i+1} - \Delta y_i}{\Delta x^2} \tag{B.14}$$

By applying once more the right derivative definition we get $\Delta y_{i+1} = y_{i+2} - y_{i+1}$ and $\Delta y_i = y_{i+1} - y_i$ and by substituting into the above equation we can obtain:

$$\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{\Delta y_{i+1} - \Delta y_i}{\Delta x^2} = \frac{(y_{i+2} - y_{i+1}) - (y_{i+1} - y_i)}{\Delta x^2} = \frac{(y_{i+2} - 2y_{i+1} + y_i)}{\Delta x^2} \tag{B.15}$$

Then, we define the operator $\Delta^2 y = y_{i+2} - 2y_{i+1} + y_i$ for the right finite difference case. Note that we only use the points on the right of the point x_i. In Figure B.4 we constructed a figure in order to obtain automatically the operators for higher order operators.

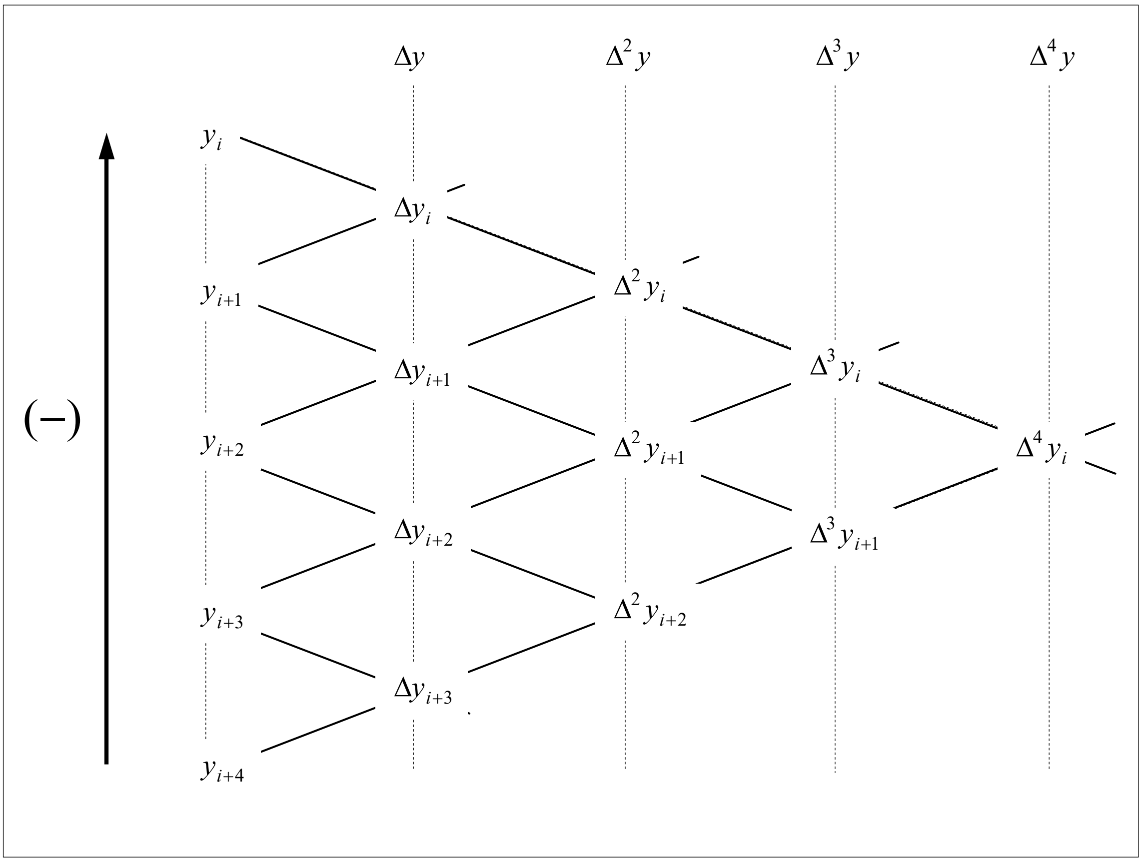

Figure B.4: Right finite differences.

For example, in order to obtain the operator $\Delta^3 y$ by means of Figure B.4 it is enough to do:

$$\begin{aligned}\Delta^3 y &= \Delta^2 y_{i+1} - \Delta^2 y_i = (\Delta y_{i+2} - \Delta y_{i+1}) - (\Delta y_{i+1} - \Delta y_i) = \Delta y_{i+2} - 2\Delta y_{i+1} + \Delta y_i \\ &= (y_{i+3} - y_{i+2}) - 2(y_{i+2} - y_{i+1}) + (y_{i+1} - y_i) \\ &= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i\end{aligned} \tag{B.16}$$

With that we can define the third derivative by means of right finite difference as follows:

$$\left(\frac{\Delta^3 y}{\Delta x^3}\right)_i = \frac{y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i}{\Delta x^3} \tag{B.17}$$

B.2.3 Central Finite Differences

The central finite difference uses simultaneously the points on the left and on the right. According to Figure B.4 we can define an automatic way in order to obtain the operators $\Delta y, \Delta^2 y, \cdots$ when we use the central finite difference.

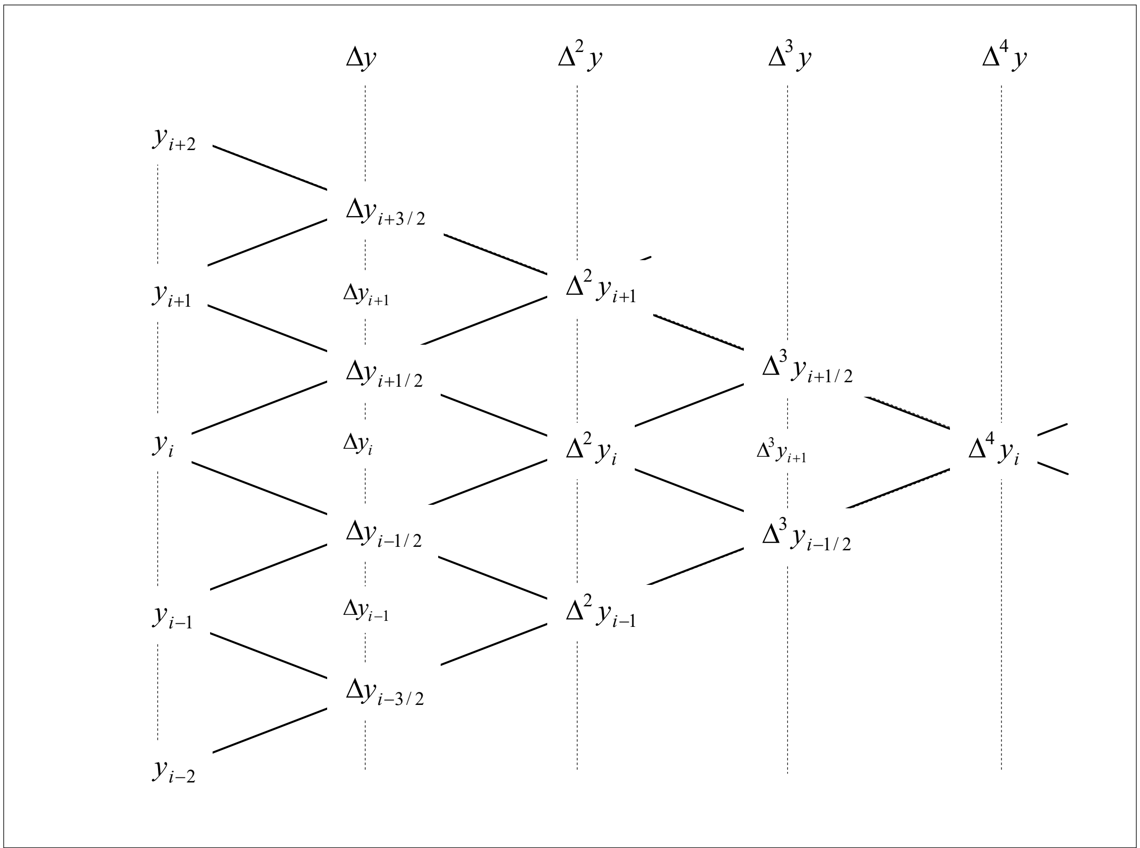

Figure B.5: Central finite differences.

In Figure B.5 the term $\Delta y_{i+3/2}$ characterizes finite difference taking at the point between x_{i+1} and x_{i+2}. For example, to obtain the first derivative, we localize the term Δy_i in Figure B.5, such term is between $\Delta y_{i+1/2}$ and $\Delta y_{i-1/2}$ and we take the average:

$$\begin{aligned}
&\Delta y_i = \frac{\Delta y_{i+1/2} + \Delta y_{i-2}}{2} = \frac{(y_{i+1} - y_i) + (y_i - y_{i-1})}{2} = \frac{y_{i+1} - y_{i-1}}{2} \\
&\Rightarrow \left(\frac{\Delta y}{\Delta x}\right)_i = \frac{y_{i+1} - y_{i-1}}{2\Delta x}
\end{aligned} \tag{B.18}$$

According to Figure B.5 we can represent the second derivative $\Delta^2 y_i = \Delta y_{i+1/2} - \Delta y_{i-2}$, then:

$$\begin{aligned}
&\Delta^2 y_i = \Delta y_{i+1/2} - \Delta y_{i-2} = (y_{i+1} - y_i) - (y_i - y_{i-1}) = y_{i+1} - 2y_i + y_{i-1} \\
&\Rightarrow \left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}
\end{aligned} \tag{B.19}$$

In the same way the third derivative can be represented as follows:

$$\begin{aligned}\Delta^3 y_i &= \frac{\Delta^3 y_{i+1/2} + \Delta^3 y_{i-2}}{2} = \frac{(\Delta^2 y_{i+1} - \Delta^2 y_i) + (\Delta^2 y_i - \Delta^2 y_{i-1})}{2} \\ &= \frac{\Delta^2 y_{i+1} - \Delta^2 y_{i-1}}{2} = \frac{[\Delta y_{i+3/2} - \Delta y_{i+1/2}] - [\Delta y_{i-1/2} - \Delta y_{i-3/2}]}{2} \\ &= \frac{[(y_{i+2} - y_{i+1}) - (y_{i+1} - y_i)] - [(y_i - y_{i-1}) - (y_{i-1} - y_{i-2})]}{2} \\ &= \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2}\end{aligned} \tag{B.20}$$

thus

$$\left(\frac{\Delta^3 y}{\Delta x^3}\right)_i = \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2\Delta x^3} \tag{B.21}$$

Notice that when we are applying the central finite differences for derivatives of odd order it appears 2 in the denominator.

NOTE: For the finite differences of even order, e.g. $\Delta^2 y, \Delta^4 y, \Delta^6 y, \cdots$, the coefficients are the same as those for the binomial expression $(a-b)^n$, for instance

$$(a-b)^2 = 1a^2 - 2ab + 1b^2 \tag{B.22}$$

and the coefficients are $(1,-2,1)$. Another example

$$(a-b)^4 = 1a^4 - 4a^3 b + 6a^2 b^2 - 4ab^3 + 1b^4 \tag{B.23}$$

and the coefficients are $(1,-4,6,-4,1)$

B.3 Finite Differences to Partial Derivatives

Let us consider now the function $z = z(x,y)$. The partial derivatives can be approached by means of Central Finite Differences as follows:

$$\begin{aligned}&\left(\frac{\partial z}{\partial x}\right)_{i,j} \approx \frac{z_{i+1,j} - z_{i-1,j}}{2\Delta x}; \qquad \frac{\partial^2 z}{\partial x^2} \approx \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\Delta x^2} \\ &\left(\frac{\partial z}{\partial y}\right)_{i,j} \approx \frac{z_{i,j+1} - z_{i,j-1}}{2\Delta y}; \qquad \frac{\partial^2 z}{\partial y^2} \approx \frac{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}}{\Delta y^2}\end{aligned} \tag{B.24}$$

with that we can also obtain:

$$\begin{aligned}&\left(\frac{\partial^2 z}{\partial y \partial x}\right)_{i,j} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \approx \frac{\partial}{\partial y}\left[\frac{z_{i+1,j} - z_{i-1,j}}{2\Delta x}\right] = \frac{1}{2\Delta x}\left[\frac{\partial}{\partial y}\left(z_{i+1,j}\right) - \frac{\partial}{\partial y}\left(z_{i-1,j}\right)\right] \\ &\left(\frac{\partial^2 z}{\partial y \partial x}\right)_{i,j} \approx \frac{1}{4\Delta x \Delta y}\left(z_{i+1,j+1} - z_{i+1,j-1} - z_{i-1,j+1} + z_{i-1,j-1}\right)\end{aligned} \tag{B.25}$$

If we consider the domain discretized into points, (see Figure B.6), we can represent the partial derivative $\left(\frac{\partial^2 z}{\partial y \partial x}\right)_{i,j}$ in operator form as indicated in Figure B.7, in which $\Delta x = h$, $\Delta y = k$.

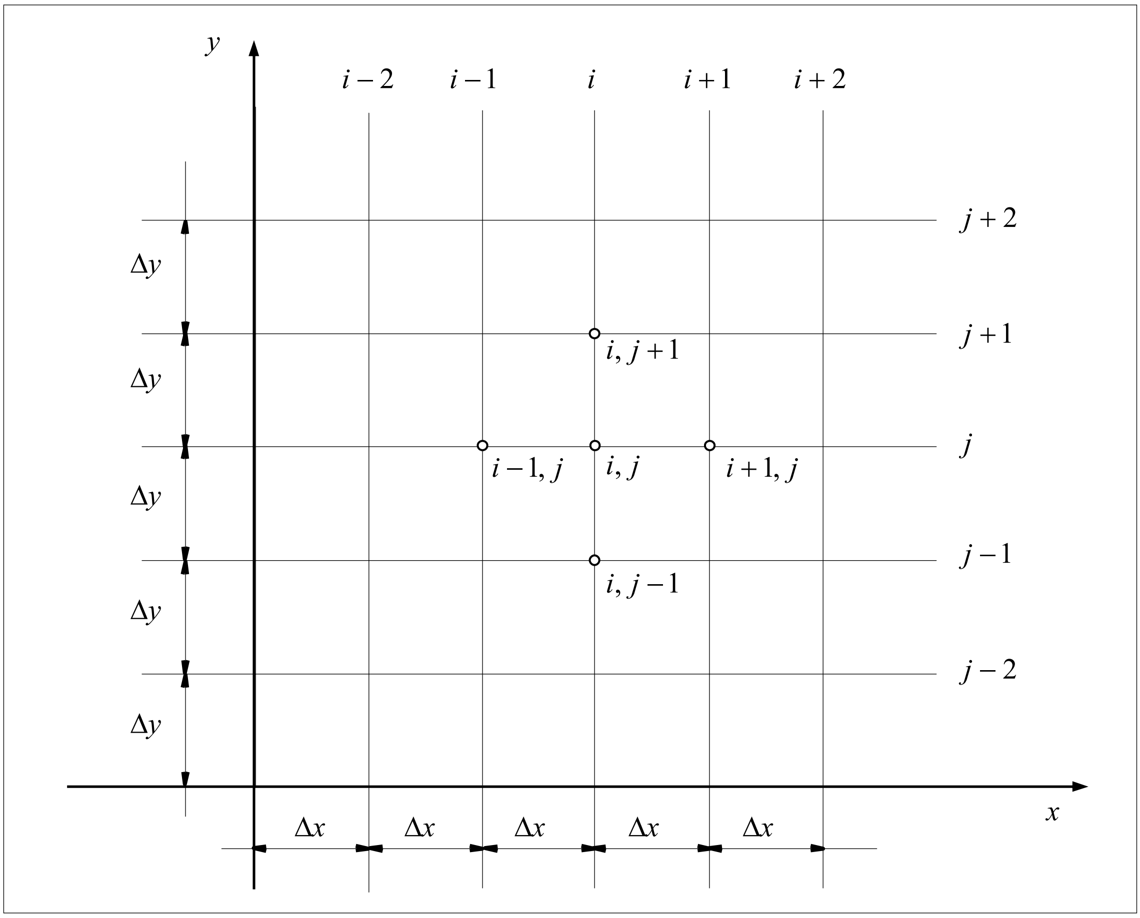

Figure B.6: Domain discretization.

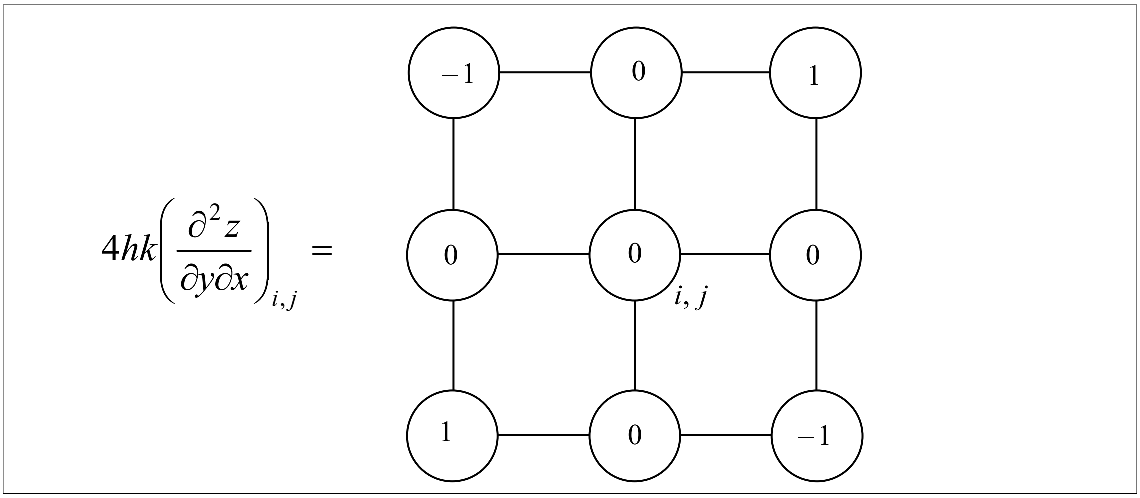

Figure B.7

In the same fashion we can represent

$$
\begin{aligned}
&\left(\frac{\partial^4 z}{\partial y^2 \partial x^2}\right) = \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^2}{\partial y^2}\left[\frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{h^2}\right] \\
&\Rightarrow \left(\frac{\partial^4 z}{\partial y^2 \partial x^2}\right) = \frac{1}{h^2 k^2}\left(\begin{aligned} &z_{i+1,j+1} - 2z_{i+1,j} + z_{i+1,j-1} - 2z_{i,j+1} \\ &\qquad + 4z_{i,j} - 2z_{i,j-1} + z_{i-1,j+1} - 2z_{i-1,j} + z_{i-1,j-1} \end{aligned}\right)
\end{aligned}
\tag{B.26}
$$

The above equation can be represented if form of operator as the one indicated in Figure B.8.

$$4h^2k^2\left(\frac{\partial^4 z}{\partial y^2 \partial x^2}\right)_{i,j} =$$

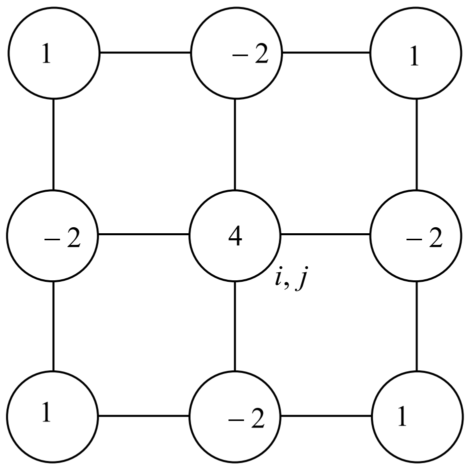

Figure B.8

As we have seen previously the following is true $\left(\frac{\Delta^2 y}{\Delta x^2}\right)_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$, then, the partial derivative can be written as follows:

$$\left(\frac{\partial^2 z}{\partial x^2}\right)_{i,j} = \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\Delta x^2} \tag{B.27}$$

In the same fashion we can obtain

$$\left(\frac{\partial^2 z}{\partial y^2}\right)_{i,j} = \frac{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}}{\Delta y^2} \tag{B.28}$$

with that, the Laplacian $\nabla^2 z$ becomes:

$$\nabla^2 z = \left(\frac{\partial^2 z}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 z}{\partial y^2}\right)_{i,j} \approx \frac{z_{i,j+1} - 2z_{i,j} + z_{i,j-1}}{\Delta y^2} + \frac{z_{i+1,j} - 2z_{i,j} + z_{i-1,j}}{\Delta x^2} \tag{B.29}$$

Example

Let us consider the following partial differential equation

$$\nabla^2 z = -\frac{p}{S} \tag{B.30}$$

where z represents the membrane deflection under the pressure p, in which the membrane deflection on the boundary is zero. The square cross section has side $b = 6h$ as indicated in Figure B.9. Obtain the membrane deflection z in the cross section.

Solution:

We can use the symmetry of the cross section and analyze only a quarter of the section. Besides, in this quarter there are points with the same displacement, with that we will need to analyze only the half of the quarter, (see Figure B.9).

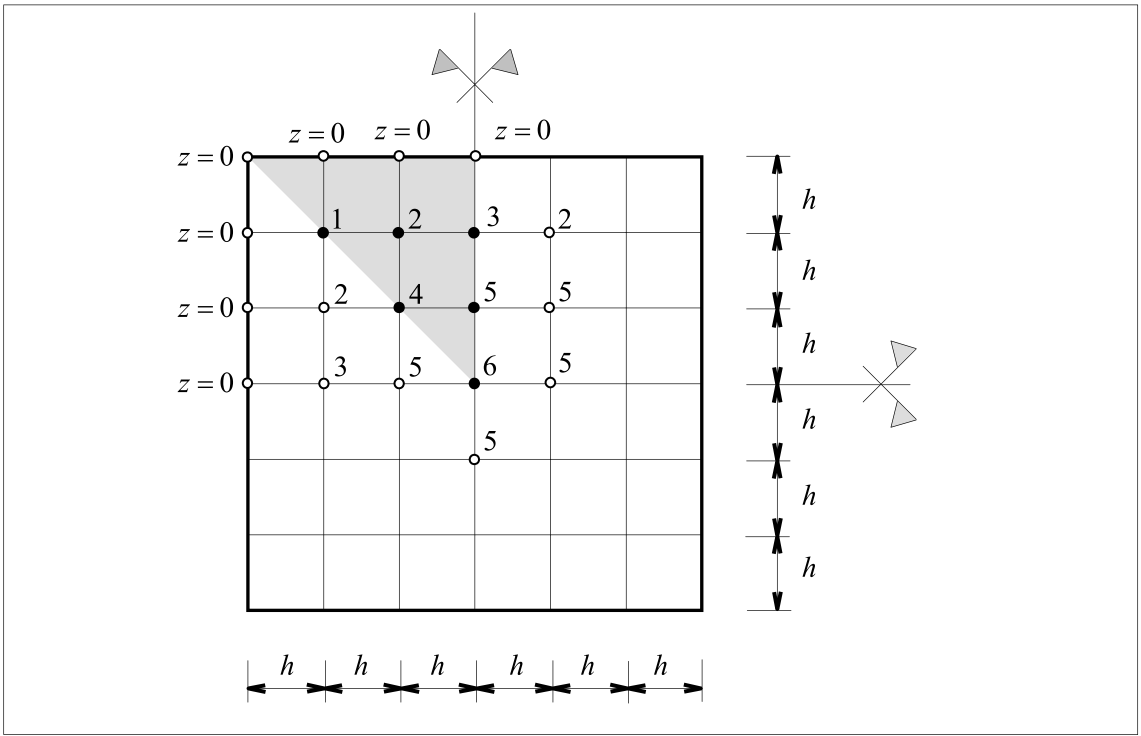

Figure B.9: Discretization of the domain.

As seen previously, the Laplacian can be approached by means of the finite difference:

$$\nabla^2 z \approx z_{i,j+1} + z_{i,j-1} + z_{i+1,j} + z_{i-1,j} - 4z_{i,j} = \frac{-h^2 p}{S} \tag{B.31}$$

where we have considered $\Delta x^2 = \Delta y^2 = h^2$. The operator can be appreciated in Figure B.10.

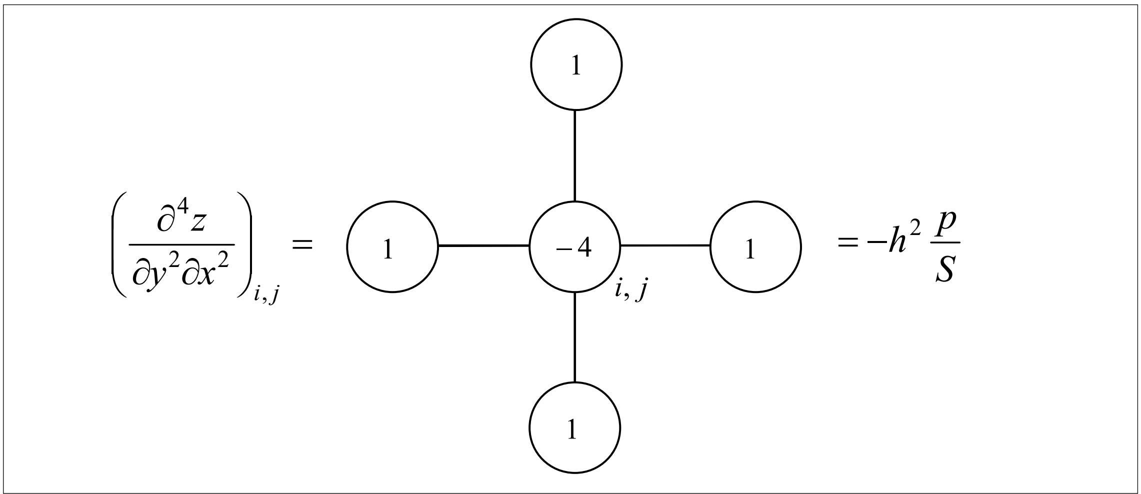

Figure B.10

By applying the operator described in Figure B.10 to the points of the mesh $(1,2,\cdots,6)$, (see Figure B.9), we can construct the following set of equations:

$$\begin{bmatrix} -4z_1 & +2z_2 & & & & \\ z_1 & -4z_2 & +z_3 & +z_4 & & \\ & +2z_2 & -4z_3 & & +z_5 & \\ & +2z_2 & & -4z_4 & +2z_5 & \\ & & z_3 & +2z_4 & -4z_5 & +z_6 \\ & & & & 4z_5 & -4z_6 \end{bmatrix} = \frac{-h^2 p}{S} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{B.32}$$

By restructuring the above set of equations we can obtain:

$$\begin{bmatrix} -4 & 2 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 1 & 0 & 0 \\ 0 & 2 & -4 & 0 & 1 & 0 \\ 0 & 2 & 0 & -4 & 2 & 0 \\ 0 & 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 4 & -4 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{Bmatrix} = \frac{-h^2 p}{S} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} \tag{B.33}$$

By solving the above set of equations we can obtain:

$$\begin{aligned} z_1 = 0.95192\frac{h^2 p}{S} \quad ; \quad z_2 = 1.4035\frac{h^2 p}{S} \quad ; \quad z_3 = 1.53846\frac{h^2 p}{S} \\ z_4 = 2.1250\frac{h^2 p}{S} \quad ; \quad z_5 = 2.34615\frac{h^2 p}{S} \quad ; \quad z_6 = 2.59615\frac{h^2 p}{S} \end{aligned} \tag{B.34}$$