Annex A

Numerical Integration over Time

A.1 Introduction

Before raising the case for multiple degrees of freedom we will analysis the numerical solution for the problem:

$$y'(x,t) = \frac{dy(x,t)}{dt} \tag{A.1}$$

which goal is to find the function $y(x,t)$.

The partial differential exact solution for the most engineering problems cannot be obtained due to the problem complexity. But, for a better understanding of the procedure numerical solution we will adopt the very simply differential equation, (see Chapra&Canale (1988)):

$$y'(t) = \frac{dy(t)}{dt} = -2t^3 + 12t^2 - 20t + 8.5 \tag{A.2}$$

which is time dependent only. The exact solution can be obtained by integrate the above equation over time, i.e.:

$$y = -0.5t^4 + 4t^3 - 10t^2 + 8.5t + C \tag{A.3}$$

where C is the constant of integration. Note that there are infinite solutions, since C could assume infinite values, (see Figure A.1). The solution is unique if the initial condition is known. For this example (A.2) which is only time dependent the function value at $t = 0$ is known and given by $y(t=0) \equiv y_0 = 1 \Rightarrow C = 1$. Then, the solution is unique:

$$y = -0.5t^4 + 4t^3 - 10t^2 + 8.5t + 1 \tag{A.4}$$

At $t = 0$ we know the following parameters:

$$
\begin{cases} y_0 = 1 \\ y'_0 = 8.5 \end{cases} \tag{A.5}
$$

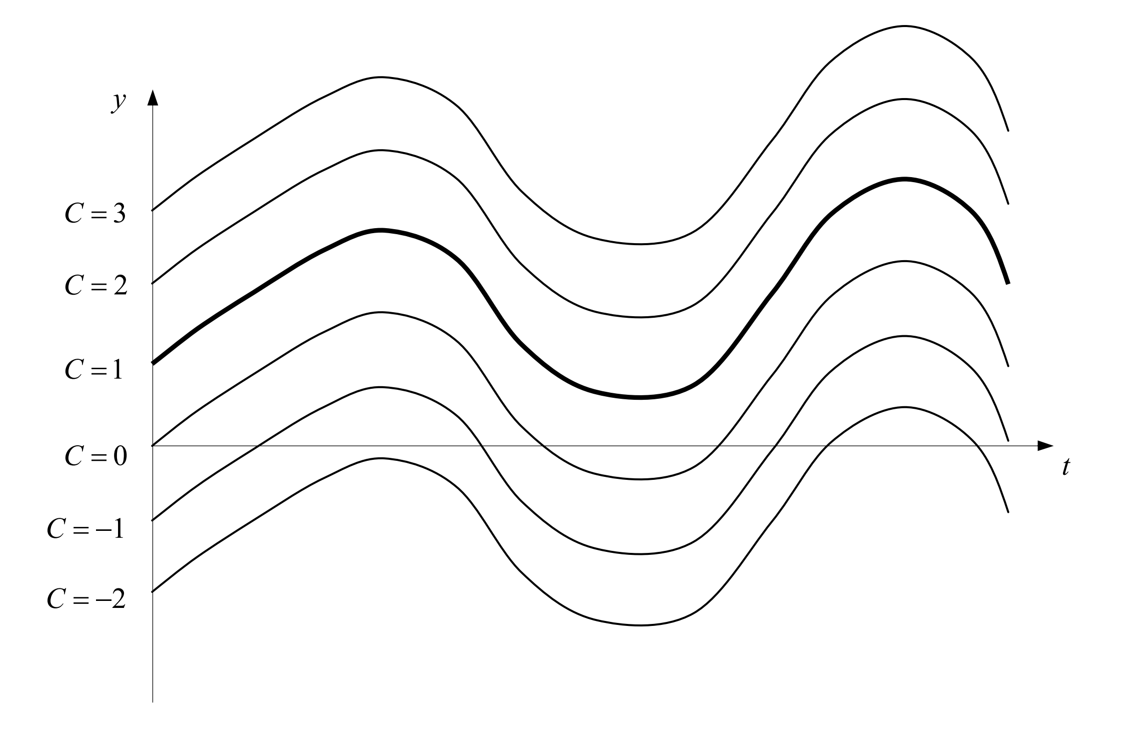

Figure A.1: Infinites solutions.

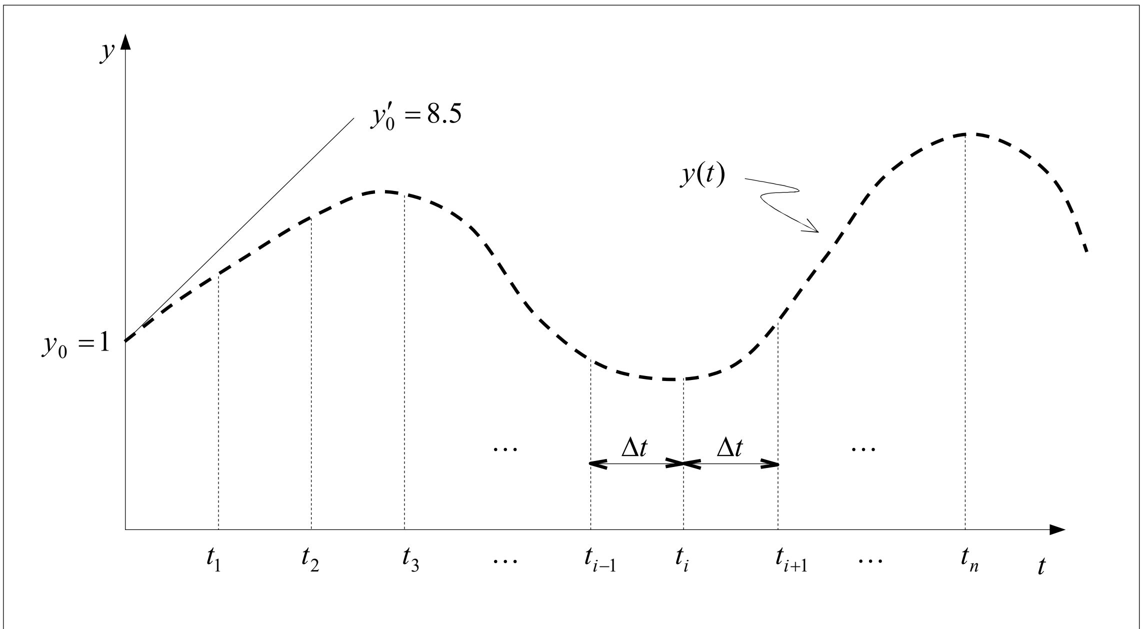

Figure A.2: Time discretization.

One of the most techniques used for numerical integration over time is the Finite Difference Method in which the "domain" is discretized by the finite value Δt (time increment). Next we will discuss some of these methods.

A.2 Euler's Method

Knowing the value of the curve slope at time t, i.e. y_i', we can obtain the next approximated value for y_{i+1} by means of a lineal approach:

$$y_i' = \frac{y_{i+1} - y_i}{\Delta t} \quad \Rightarrow \quad y_{i+1} = y_i + y_i' \Delta t \tag{A.6}$$

The above approach is the same as forward finite difference.

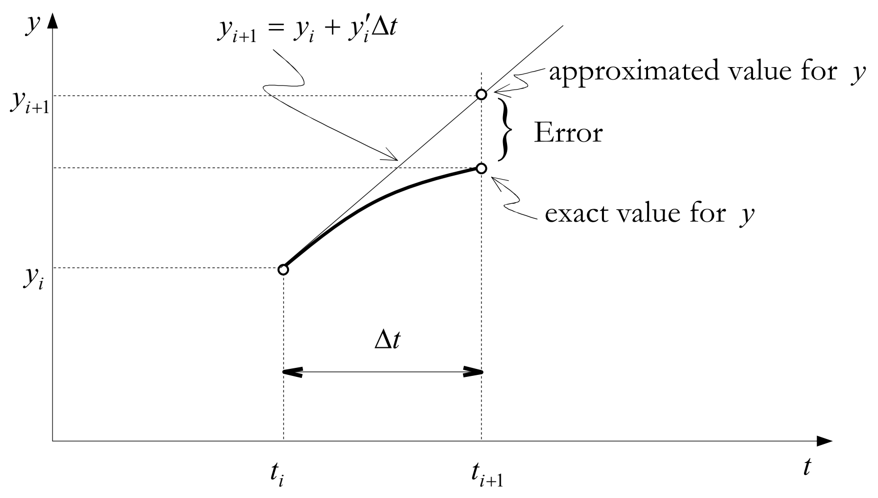

Figure A.3: Forward Finite Difference.

By means of Figure A.3 we can guess that when the time increment tends to zero we approach the exact value of the function. For problems with several unknowns working with very small time increment it can result in a high computational cost, so, to overcome this drawback some effective methods have been developed in order to guarantee result accuracy even when the time increment is big.

In previous example we have applied the forward finite difference using y_i' to obtain y_{i+1}. We can also use the following approach: we situate at y_{i+1} and we apply the backward finite difference, i.e.:

$$y_{i+1}' = \frac{y_{i+1} - y_i}{\Delta t} \quad \Rightarrow \quad y_{i+1} = y_i + y_{i+1}' \Delta t \tag{A.7}$$

This method is known as backward Euler's method (implicit method). With that we can summarize that:

$$y_{i+1} = y_i + y_i' \Delta t \quad \text{(Explicit method)} \tag{A.8}$$

$$y_{i+1} = y_i + y'_{i+1}\Delta t \quad \text{(Implicit method)} \tag{A.9}$$

Another approach we can adopt is by consider the curve slope as the average between y'_i and y'_{i+1}, i.e.:

$$y_{i+1} = y_i + \left(\frac{y'_i + y'_{i+1}}{2}\right)\Delta t \tag{A.10}$$

which method is more precise than forward/backward finite difference. This method is called Crank-Nicolson's method.

A.3 Alfa Method

We can generalize the above method in a single expression. To do this we consider the following approaches to the functions $y(t)$ and $y'(t)$, (see Figure A.4).

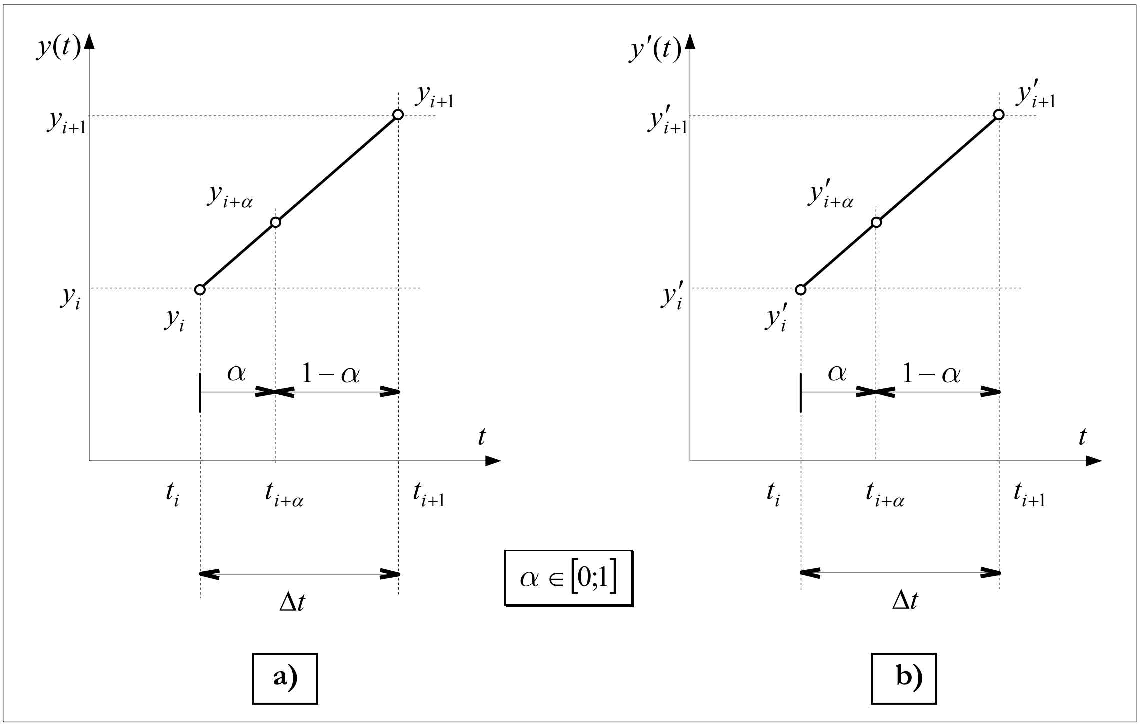

Figure A.4: Alfa method.

By means of linearization for the functions $y(t)$ and $y'(t)$, we can obtain:

$$y'_{i+\alpha} = \frac{y_{i+1} - y_i}{\Delta t} \tag{A.11}$$

with that we can conclude that:

$$y_{i+1} = y_i + y'_{i+\alpha}\Delta t \tag{A.12}$$

Using similarity triangle, (see Figure A.4(b)), it is possible to express $y'_{i+\alpha}$ as follows:

$$\frac{y'_{i+1} - y'_i}{1} = \frac{y'_{i+\alpha} - y'_i}{\alpha} \quad \Rightarrow \quad y'_{i+\alpha} = y'_i + \alpha\left(y'_{i+1} - y'_i\right) \tag{A.13}$$

or:

$$y'_{i+\alpha} = \alpha\, y'_{i+1} + (1-\alpha) y'_i \tag{A.14}$$

The we summarize the Alfa method as follows:

$$\boxed{\begin{cases} y_{i+1} = y_i + y'_{i+\alpha}\Delta t \\ y'_{i+\alpha} = \alpha\, y'_{i+1} + (1-\alpha) y'_i \end{cases}} \tag{A.15}$$

in which depending on the α value we obtain::

- $\alpha = 0$ (Explicit)

$$\begin{cases} y_{i+1} = y_i + y'_{i+\alpha}\Delta t \\ y'_{i+\alpha} = (1) y'_i \end{cases} \quad \Rightarrow \quad y_{i+1} = y_i + y'_i \Delta t \tag{A.16}$$

- $\alpha = 1$ (Implicit)

$$\begin{cases} y_{i+1} = y_i + y'_{i+\alpha}\Delta t \\ y'_{i+\alpha} = y'_{i+1} \end{cases} \quad \Rightarrow \quad y_{i+1} = y_i + y'_{i+1} \Delta t \tag{A.17}$$

- $\alpha = \frac{1}{2}$ (Crank-Nicolson)

$$\begin{cases} y_{i+1} = y_i + y'_{i+\frac{1}{2}}\Delta t \\ y'_{i+\frac{1}{2}} = \frac{1}{2} y'_{i+1} + \left(1 - \frac{1}{2}\right) y'_i = \frac{y'_{i+1} + y'_i}{2} \end{cases} \quad \Rightarrow \quad y_{i+1} = y_i + \left(\frac{y'_{i+1} + y'_i}{2}\right)\Delta t \tag{A.18}$$

The Crank-Nicolson's method is also called Heun's method. Geometrically we can interpret as indicated in Figure A.5. At t_i we make a prediction for $y^0_{i+1} = y_i + y'_i \Delta t$ and in turn we can obtain y'^0_{i+1}. Then we find the value for the new curve slope:

$$\bar{y}'_i = \frac{y'^0_{i+1} + y'_i}{2} \tag{A.19}$$

Then we obtain once again the new value for y_{i+1} by considering the slope $\bar{y}'_i$:

$$y_{i+1} = y_i + \left(\frac{y'^0_{i+1} + y'_i}{2}\right)\Delta t \tag{A.20}$$

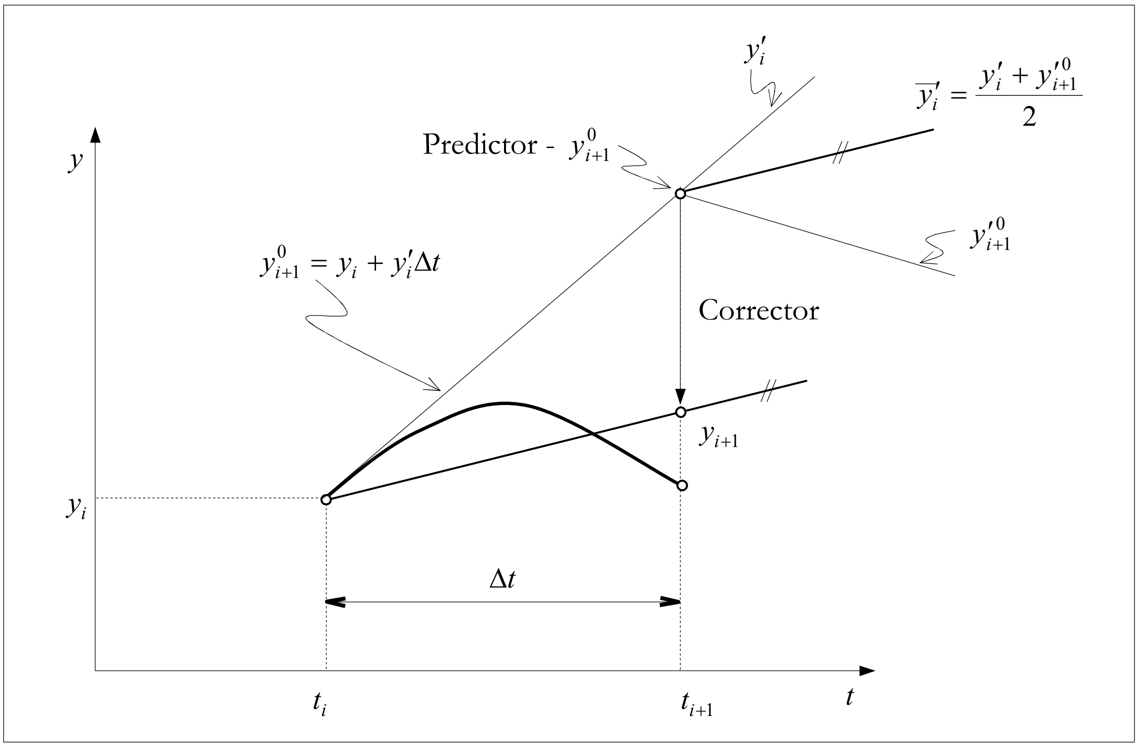

Figure A.5: Heun's method (predictor-corrector).

Returning to our example initially raised whose differential equations is:

$$y' = \frac{dy}{dt} = -2t^3 + 12t^2 - 20t + 8.5 \tag{A.21}$$

with the initial condition:

$$t = 0 \quad ; \quad y_0 = 1 \quad \Rightarrow \quad y'_0 = 8.5 \tag{A.22}$$

We will aplly the Euler's method and the Heun's method with time increment $\Delta t = 0.5s$. For the first time step $(t = 0.5)$ the exact value of the function can be obtained by means of (A.4), i.e.:

$$y(t = 0.5) = -0.5 \times 0.5^4 + 4 \times 0.5^3 - 10 \times 0.5^2 + 8.5 \times 0.5 + 1 = 3.21875 \tag{A.23}$$

The numerical procedure follows:

Predictor $i+1$

$y_1^0 = y_0 + y'_0 \Delta t = 1 + 8.5 \times 0.5 = 5.25$ (We stop here if we are using the Euler's method)

Corrector

$$y'(t = 0.5) = y'^0_1 = -2t^3 + 12t^2 - 20t + 8.5 = -2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5 = 1.25$$

$$\overline{y}'_i = \frac{y'^0_{i+1} + y'_i}{2} = \frac{1.25 + 8.5}{2} = 4.875$$

$$y_1 = y_0 + \overline{y}'_0 \Delta t = 1 + 4.875 \times 0.5 = 3.4375$$

A.4 Modified Euler's Method

In the modified Euler's method we use the Euler's method to predict the value function in the middle of the interval:

$$y_{i+\frac{1}{2}} = y_i + y_i' \frac{\Delta t}{2} \tag{A.24}$$

Next, we obtain the slope $y'_{i+\frac{1}{2}}$ at this point, which is used to obtain y_{i+1}:

$$y_{i+1} = y_i + y'_{i+\frac{1}{2}} \Delta t \tag{A.25}$$

Apply this methodology to the proposed example (A.21) we can obtain:

$$y_{i+\frac{1}{2}} = y_i + y_i' \frac{\Delta t}{2} = 1 + 8.5 \times \frac{0.5}{2} = 3.125$$

Slope calculation at the middle point:

$$y'(t = 0.25) = y'_{0+\frac{1}{2}} = -2t^3 + 12t^2 - 20t + 8.5 = -2(0.25)^3 + 12(0.25)^2 - 20(0.25) + 8.5 = 4.21875$$

$$y_1 = y_0 + y'_{\frac{1}{2}} \Delta t = 1 + 4.21875 \times 0.5 = 3.1093$$

In Figure A.6 we can appreciate this procedure to calculate the function, $y'(t) = \frac{dy(t)}{dt} = -2t^3 + 12t^2 - 20t + 8.5$ with $y'(0) = 1$, in which we have used the Euler's method, modified Euler's method and the Heun's method with time increment $\Delta t = 0.5$.

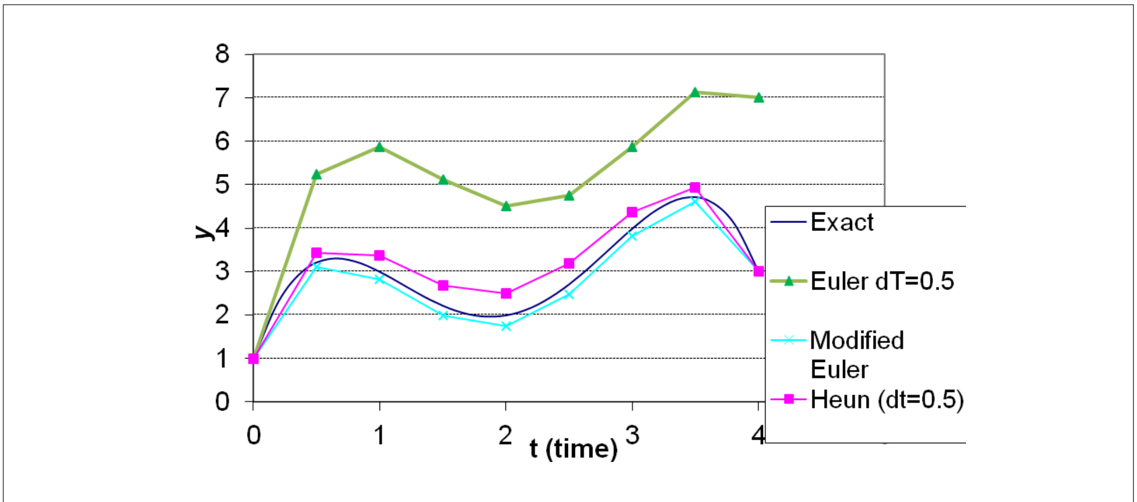

Figure A.6: Comparative responses between some methods.

As we can see the Heun's method (H) always overestimates the function value while the Modified Euler's method (ME) underestimates the function value. We can make the following approximation for y_{i+1}:

$$y_{i+1} = \frac{1}{3}\left(y_{i+1}^{H} + 2 y_{i+1}^{ME}\right) \tag{A.26}$$

By substituting the equations (A.20) and (A.25) into the above equation we can obtain:

$$
\begin{aligned}
y_{i+1} &= \frac{1}{3}\left(y_{i+1}^{H} + 2y_{i+1}^{ME}\right) \\
&= \frac{1}{3}\left[y_i + \left(\frac{y_{i+1}^{\prime 0} + y_i'}{2}\right)\Delta t + 2\left(y_i + y'_{i+\frac{1}{2}}\Delta t\right)\right] \\
&= \frac{1}{3}\left[y_i + \frac{y_{i+1}^{\prime 0}}{2}\Delta t + \frac{y_i'}{2}\Delta t + 2y_i + 2y'_{i+\frac{1}{2}}\Delta t\right] \\
&= \frac{1}{3}\left[3y_i + \frac{y_{i+1}^{\prime 0}}{2}\Delta t + \frac{y_i'}{2}\Delta t + 2y'_{i+\frac{1}{2}}\Delta t\right]
\end{aligned}
\tag{A.27}
$$

thus:

$$
y_{i+1} = y_i + \frac{\Delta t}{6}\left[y_i' + 4y'_{i+\frac{1}{2}} + y_{i+1}^{\prime 0}\right] \tag{A.28}
$$

whose equation is known as Runge-Kutta's integration method of third order (see Chapra&Canale (1988)), which is a good approximation as we can see in Figure A.7.

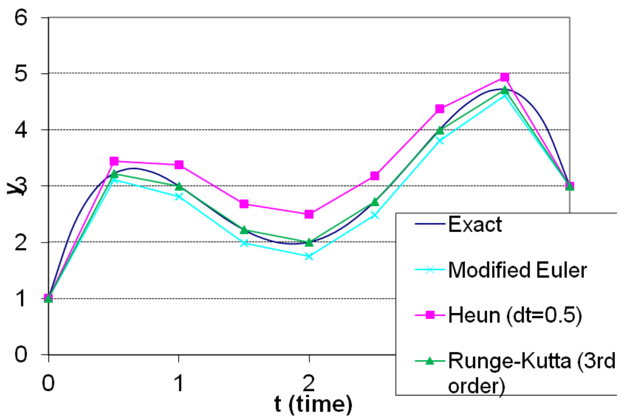

Figure A.7: Comparative responses between some methods (Runge-Kutta).

A.5 Unsteady Case with Multiply Degree-of-Freedom

Let us consider the following set of equations:

$$\boldsymbol{D}\,\dot{\boldsymbol{T}} + \boldsymbol{K}\,\boldsymbol{T} = \boldsymbol{F} \tag{A.29}$$

For the thermal problem, $\boldsymbol{D}$ stands for capacitance matrix, $\boldsymbol{K}$ is the conductivity matrix, and $\boldsymbol{T}$ represents nodal temperature values.

By considering that

$$\dot{\boldsymbol{T}} = \frac{\boldsymbol{T}_{t+1} - \boldsymbol{T}_t}{\Delta t} \tag{A.30}$$

and by apply the Alfa method:

$$\begin{aligned} \boldsymbol{T}_\alpha &= \alpha \boldsymbol{T}_{t+1} + (1-\alpha)\boldsymbol{T}_t \\ \boldsymbol{F}_\alpha &= \alpha \boldsymbol{F}_{t+1} + (1-\alpha)\boldsymbol{F}_t \end{aligned} \tag{A.31}$$

By substituting the equations (A.30) and (A.31) into the equation (A.29), we can obtain:

$$\boldsymbol{D}\left(\frac{\boldsymbol{T}_{t+1} - \boldsymbol{T}_t}{\Delta t}\right) + \boldsymbol{K}\,\boldsymbol{T}_\alpha = \boldsymbol{F}_\alpha \tag{A.32}$$

$$\boldsymbol{D}\left(\frac{\boldsymbol{T}_{t+1} - \boldsymbol{T}_t}{\Delta t}\right) + \boldsymbol{K}\left[\alpha \boldsymbol{T}_{\alpha+1} + (1-\alpha)\boldsymbol{T}_\alpha\right] = \alpha \boldsymbol{F}_{\alpha+1} + (1-\alpha)\boldsymbol{F}_\alpha \tag{A.33}$$

then:

$$\left[\frac{\boldsymbol{D}}{\Delta t} + \alpha \boldsymbol{K}\right]\boldsymbol{T}_{t+1} = \alpha\,\boldsymbol{F}_{t+1} + (1-\alpha)\boldsymbol{F}_t + \left[\frac{\boldsymbol{D}}{\Delta t} - (1-\alpha)\boldsymbol{K}\right]\boldsymbol{T}_t \tag{A.34}$$

$$\boldsymbol{K}^{eff}\,\boldsymbol{T}_{t+1} = \boldsymbol{F}^{eff} \tag{A.35}$$

where

$$\boldsymbol{K}^{eff} = \frac{\boldsymbol{D}}{\Delta t} + \alpha \boldsymbol{K} \quad ; \quad \boldsymbol{F}^{eff} = \alpha\,\boldsymbol{F}_{t+1} + (1-\alpha)\boldsymbol{F}_t + \left[\frac{\boldsymbol{D}}{\Delta t} - (1-\alpha)\boldsymbol{K}\right]\boldsymbol{T}_t \tag{A.36}$$

A.6 Dynamic Analysis with Numerical Integration

The more general approach to the solution of the dynamic response for structures is the direct numerical integration of the dynamic equilibrium:

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{D}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \tag{A.37}$$

whose equations must fulfill for all time t, then it is also valid at time $t + \Delta t$:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} \tag{A.38}$$

where $\boldsymbol{M}$ is the mass matrix, $\boldsymbol{D}$ is the damping matrix, $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{F}$ is the nodal external force vector, and $\boldsymbol{U}$, $\dot{\boldsymbol{U}}$, $\ddot{\boldsymbol{U}}$ are displacement, velocity and acceleration, respectively. The absence of subscript time step in the matrices $\boldsymbol{M}$, $\boldsymbol{D}$ and $\boldsymbol{K}$ indicates a linear problem, i.e. they do not depend on $\boldsymbol{U}$, $\dot{\boldsymbol{U}}$ and $\ddot{\boldsymbol{U}}$. In the case in which the structure presents a material non-linearity the matrix $\boldsymbol{K}$ depends on $\boldsymbol{U}$.

For a system without damping ($\boldsymbol{D} = \boldsymbol{0}$) the energy is conserved (system without energy dissipation), and the sum of the internal energy ($\frac{1}{2}\dot{\boldsymbol{U}}^{\boldsymbol{T}}\boldsymbol{M}\dot{\boldsymbol{U}}$) plus the strain energy ($\frac{1}{2}\boldsymbol{U}^{\boldsymbol{T}}\boldsymbol{K}\boldsymbol{U}$) is constant at any time step:

$$2E = \dot{\boldsymbol{U}}_t^T \boldsymbol{M}\dot{\boldsymbol{U}}_t + \boldsymbol{U}_t^T \boldsymbol{K}\boldsymbol{U}_t = \dot{\boldsymbol{U}}_{t+\Delta t}^T \boldsymbol{M}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{U}_{t+\Delta t}^T \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} \tag{A.39}$$

The numerical analysis of the dynamic system (A.37) can be inefficient where $\boldsymbol{D}$ is responsible for the damping (energy dissipation) in the structure. Therefore some methods were developed in order to introduce a numerical damping (artificial) which generally is controlled by a parameter. For example, we can replace the damping matrix $\boldsymbol{D}$ by a linear combination between $\boldsymbol{K}$ and $\boldsymbol{M}$ (Rayleigh damping):

$$\boldsymbol{D} = \alpha\boldsymbol{K} + \beta\boldsymbol{M} \tag{A.40}$$

Then, we will study some methods in which a numerical damping is introduced albeit the equation has no matrix $\boldsymbol{D}$.

Several numerical techniques have been developed for solving the set of equations (A.38). We can classify these techniques as *Explicit*, *Implicit*, or *Mixed*.

The Explicit methods do not require information at the time step $t + \Delta t$ to predict the response at time $t + \Delta t$, i.e.:

$$\boldsymbol{U}_{t+\Delta t} = f\left(\boldsymbol{U}_t, \dot{\boldsymbol{U}}_t, \ddot{\boldsymbol{U}}_t, \dot{\boldsymbol{U}}_{t-\Delta t}, \cdots\right) \tag{A.41}$$

These methods are *conditionally stable*, which implies that the time step size (Δt) must be less than a critical value (Δt_{cr}), otherwise the solution is not stable, i.e. the solution diverges.

The Implicit methods use the information at time step $t + \Delta t$ to predict the structural response at time $t + \Delta t$, i.e.:

$$\boldsymbol{U}_{t+\Delta t} = f\left(\boldsymbol{U}_t, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \cdots\right) \tag{A.42}$$

With these methods it is possible to use larger time steps than those used in the explicit method. The implicit method can be unconditionally or conditionally stable. In general, the implicit methods are unconditionally stable, and the only restriction for time step size is the solution accuracy.

A.6.1 Newmark's Family of Methods

Newmark in 1959 introduced a family of integration methods for solving dynamic structural problems. To illustrate these methods we will start from the following set of equations:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} \tag{A.43}$$

We can apply the Taylor series to approximate the functions $\boldsymbol{U}$ and $\dot{\boldsymbol{U}}$:

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2!}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^3}{3!}\dddot{\boldsymbol{U}}_t + \cdots \tag{A.44}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2!}\dddot{\boldsymbol{U}}_t + \cdots \tag{A.45}$$

Newmark truncated the previous equations as follows:

$$\boldsymbol{U}_{t+\Delta t} \approx \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \beta\Delta t^3\dddot{\boldsymbol{U}}_t \tag{A.46}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} \approx \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t^2\dddot{\boldsymbol{U}}_t \tag{A.47}$$

Assuming that the acceleration varies linearly within the range $[t, t+\Delta t]$, we can apply the finite difference to approach $\dddot{\boldsymbol{U}}_t$, i.e.:

$$\dddot{\boldsymbol{U}}_t = \frac{\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t}{\Delta t} \tag{A.48}$$

Substituting the equation (A.48) into the equations (A.47) and (A.46) we can obtain:

$$\begin{aligned}\boldsymbol{U}_{t+\Delta t} &\approx \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \beta\Delta t^3\frac{\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t}{\Delta t}\\ &\approx \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \beta\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t} - \beta\Delta t^2\ddot{\boldsymbol{U}}_t\\ &\approx \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \left(\frac{1}{2} - \beta\right)\Delta t^2\ddot{\boldsymbol{U}}_t + \beta\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t}\end{aligned} \tag{A.49}$$

$$\begin{aligned}\dot{\boldsymbol{U}}_{t+\Delta t} &\approx \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t^2\frac{\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t}{\Delta t}\\ &\approx \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t} - \gamma\Delta t\ \ddot{\boldsymbol{U}}_t\\ &\approx \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t}\end{aligned} \tag{A.50}$$

Then, we summarize the approaches used by Newmark for displacement, velocity and acceleration:

$$\boxed{\begin{aligned}&\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \left(\frac{1}{2} - \beta\right)\Delta t^2\ddot{\boldsymbol{U}}_t + \beta\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t}\\ &\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t}\\ &\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}\end{aligned}} \quad \textit{Newmark's method} \tag{A.51}$$

This method is unconditionally stable when:

$$2\beta \geq \gamma \geq \frac{1}{2} \tag{A.52}$$

Solving for $\ddot{\boldsymbol{U}}_{t+\Delta t}$ by using the displacement given by the equation (A.51) we can obtain:

$$\begin{aligned}&\beta\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t} = \boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t - \Delta t\dot{\boldsymbol{U}}_t - \left(\frac{1}{2}-\beta\right)\Delta t^2\ddot{\boldsymbol{U}}_t \\ &\Rightarrow \ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\beta\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{1}{\beta\Delta t}\dot{\boldsymbol{U}}_t + \left(1-\frac{1}{2\beta}\right)\ddot{\boldsymbol{U}}_t\end{aligned} \qquad \text{(A.53)}$$

Substituting the equation (A.53) into the velocity equation (A.51) we can obtain:

$$\begin{aligned}\dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t} \\ &= \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t\ddot{\boldsymbol{U}}_t + \gamma\Delta t\left[\frac{1}{\beta\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{1}{\beta\Delta t}\dot{\boldsymbol{U}}_t + \left(1-\frac{1}{2\beta}\right)\ddot{\boldsymbol{U}}_t\right] \\ &= \frac{\gamma}{\beta\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \left(1-\frac{\gamma}{\beta}\right)\dot{\boldsymbol{U}}_t + \left(1-\frac{\gamma}{2\beta}\right)\Delta t\ddot{\boldsymbol{U}}_t\end{aligned} \qquad \text{(A.54)}$$

Thus

$$\begin{cases}\ddot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{1}{\beta\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dfrac{1}{\beta\Delta t}\dot{\boldsymbol{U}}_t + \left(1-\dfrac{1}{2\beta}\right)\ddot{\boldsymbol{U}}_t \\ \dot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{\gamma}{\beta\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \left(1-\dfrac{\gamma}{\beta}\right)\dot{\boldsymbol{U}}_t + \left(1-\dfrac{\gamma}{2\beta}\right)\Delta t\ddot{\boldsymbol{U}}_t\end{cases} \qquad \text{(A.55)}$$

By substituting the equations given by (A.55) into the equation (A.43) we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff} \qquad \text{(A.56)}$$

where

$$\begin{aligned}\boldsymbol{K}^{eff} &= \left[\frac{1}{\beta\Delta t^2}\boldsymbol{M} + \frac{\gamma}{\beta\Delta t}\boldsymbol{D} + \boldsymbol{K}\right] \\ \boldsymbol{F}^{eff} &= \boldsymbol{F}_{t+\Delta t} + \left[\frac{\boldsymbol{M}}{\beta\Delta t^2} + \frac{\gamma}{\beta\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{\boldsymbol{M}}{\beta\Delta t} - \left(1-\frac{\gamma}{\beta}\right)\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t - \left[\left(1-\frac{1}{2\beta}\right)\boldsymbol{M} + \left(1-\frac{\gamma}{2\beta}\right)\Delta t\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t\end{aligned}$$

(A.57)

It is also possible to express the above equations as follows:

$$\begin{aligned}\boldsymbol{K}^{eff} &= \left[b_1\boldsymbol{M} + b_4\boldsymbol{D} + \boldsymbol{K}\right] \\ \boldsymbol{F}^{eff} &= \boldsymbol{F}_{t+\Delta t} + \boldsymbol{M}\left[b_1\boldsymbol{U}_t - b_2\dot{\boldsymbol{U}}_t - b_3\ddot{\boldsymbol{U}}_t\right] + \boldsymbol{D}\left[b_4\boldsymbol{U}_t - b_5\dot{\boldsymbol{U}}_t - b_6\ddot{\boldsymbol{U}}_t\right]\end{aligned} \qquad \text{(A.58)}$$

where

$$\begin{aligned}&b_1 = \frac{1}{\beta\Delta t^2} \quad ; \quad b_2 = -\frac{1}{\beta\Delta t} \quad ; \quad b_3 = 1-\frac{1}{2\beta} \\ &b_4 = \gamma\Delta t b_1 \quad ; \quad b_5 = 1+\gamma\Delta t b_2 \quad ; \quad b_6 = \Delta t\left[1-\gamma+\gamma b_3\right]\end{aligned} \qquad \text{(A.59)}$$

The velocity and displacement fields can also be expressed in terms of the parameters (A.59):

$$\begin{cases}\ddot{\boldsymbol{U}}_{t+\Delta t} = b_1\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + b_2\dot{\boldsymbol{U}}_t + b_3\ddot{\boldsymbol{U}}_t \\ \dot{\boldsymbol{U}}_{t+\Delta t} = b_4\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + b_5\dot{\boldsymbol{U}}_t + b_6\ddot{\boldsymbol{U}}_t\end{cases} \qquad \text{(A.60)}$$

Next we will apply the same methodology to solve the following system:

$$\boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} \tag{A.61}$$

Given the vector $\boldsymbol{U}_{t+\Delta t}$ and its approach by using Taylor series (A.44), in which we truncate until second order term:

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \alpha\Delta t^2\ddot{\boldsymbol{U}}_t \tag{A.62}$$

Considering the following approach for $\ddot{\boldsymbol{U}}_t$:

$$\ddot{\boldsymbol{U}}_t = \frac{\dot{\boldsymbol{U}}_{t+\Delta t} - \dot{\boldsymbol{U}}_t}{\Delta t} \tag{A.63}$$

Then, the vector $\boldsymbol{U}_{t+\Delta t}$ can be rewritten as follows:

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \alpha\Delta t^2\frac{\dot{\boldsymbol{U}}_{t+\Delta t} - \dot{\boldsymbol{U}}_t}{\Delta t} \tag{A.64}$$

And the vector $\dot{\boldsymbol{U}}_{t+\Delta t}$ becomes:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\alpha\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \frac{1}{\alpha}(\alpha - 1)\dot{\boldsymbol{U}}_t \tag{A.65}$$

By substituting the equation (A.65) into the equation (A.61) we can obtain:

$$\begin{aligned}
&\boldsymbol{D}\left[\frac{1}{\alpha\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \frac{1}{\alpha}(\alpha - 1)\dot{\boldsymbol{U}}_t\right] + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}\\
&\Rightarrow\left[\frac{1}{\alpha\Delta t}\boldsymbol{D} + \boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} + \boldsymbol{D}\left[\frac{1}{\alpha\Delta t}\boldsymbol{U}_t - \frac{1}{\alpha}(\alpha - 1)\dot{\boldsymbol{U}}_t\right]\\
&\Rightarrow\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + \frac{1}{\Delta t}\boldsymbol{D}\boldsymbol{U}_t + (1-\alpha)\boldsymbol{D}\dot{\boldsymbol{U}}_t
\end{aligned} \tag{A.66}$$

And the vector $\dot{\boldsymbol{U}}_t$ can be expressed as follows:

$$\boldsymbol{D}\dot{\boldsymbol{U}}_t + \boldsymbol{K}\boldsymbol{U}_t = \boldsymbol{F}_t \quad \Rightarrow \quad \dot{\boldsymbol{U}}_t = \boldsymbol{D}^{-1}\left(\boldsymbol{F}_t - \boldsymbol{K}\boldsymbol{U}_t\right) \tag{A.67}$$

Substituting (A.67) into (A.66) we can obtain:

$$\begin{aligned}
&\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + \frac{1}{\Delta t}\boldsymbol{D}\boldsymbol{U}_t + (1-\alpha)\boldsymbol{D}\dot{\boldsymbol{U}}_t\\
&\Rightarrow\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + \frac{1}{\Delta t}\boldsymbol{D}\boldsymbol{U}_t + (1-\alpha)\boldsymbol{D}\boldsymbol{D}^{-1}\left(\boldsymbol{F}_t - \boldsymbol{K}\boldsymbol{U}_t\right)\\
&\Rightarrow\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + \frac{1}{\Delta t}\boldsymbol{D}\boldsymbol{U}_t + (1-\alpha)\boldsymbol{F}_t - (1-\alpha)\boldsymbol{K}\boldsymbol{U}_t\\
&\Rightarrow\left[\frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \alpha\boldsymbol{F}_{t+\Delta t} + (1-\alpha)\boldsymbol{F}_t + \left[\frac{1}{\Delta t}\boldsymbol{D} - (1-\alpha)\boldsymbol{K}\right]\boldsymbol{U}_t
\end{aligned} \tag{A.68}$$

or:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff} \tag{A.69}$$

where

$$\boldsymbol{K}^{eff} = \frac{1}{\Delta t}\boldsymbol{D} + \alpha\boldsymbol{K} \quad ; \quad \boldsymbol{F}^{eff} = \alpha\boldsymbol{F}_{t+\Delta t} + (1-\alpha)\boldsymbol{F}_t + \left[\frac{1}{\Delta t}\boldsymbol{D} - (1-\alpha)\boldsymbol{K}\right]\boldsymbol{U}_t \tag{A.70}$$

We can verify that the equation (A.69) is the same equation obtained by means of Alfa method employed for unsteady temperature problem, (see equation (A.34)).

A.6.1.1 Newmark's Method Scheme

I. Initial Parameters

I.1. Construction of matrices $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Obtain the parameters:

$$b_1 = \frac{1}{\beta \Delta t^2} \quad ; \quad b_2 = -\frac{1}{\beta \Delta t} \quad ; \quad b_3 = 1 - \frac{1}{2\beta}$$

$$b_4 = \gamma \Delta t b_1 \quad ; \quad b_5 = 1 + \gamma \Delta t b_2 \quad ; \quad b_6 = \Delta t [1 - \gamma + \gamma b_3]$$

I.3. Construction of $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = [b_1 \boldsymbol{M} + b_4 \boldsymbol{D} + \boldsymbol{K}]$$

I.4. Given the boundary conditions $\boldsymbol{U}_0$, $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1} \left[\boldsymbol{F}_0 - \boldsymbol{D} \dot{\boldsymbol{U}}_0 - \boldsymbol{K} \boldsymbol{U}_0 \right]$$

I.5. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_0 \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_0$$

II. For each time step $t + \Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_{t+\Delta t} + \boldsymbol{M} \left[b_1 \boldsymbol{U}_t - b_2 \dot{\boldsymbol{U}}_t - b_3 \ddot{\boldsymbol{U}}_t \right] + \boldsymbol{D} \left[b_4 \boldsymbol{U}_t - b_5 \dot{\boldsymbol{U}}_t - b_6 \ddot{\boldsymbol{U}}_t \right]$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff} \boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\begin{cases} \ddot{\boldsymbol{U}}_{t+\Delta t} = b_1 \left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t \right) + b_2 \dot{\boldsymbol{U}}_t + b_3 \ddot{\boldsymbol{U}}_t \\ \dot{\boldsymbol{U}}_{t+\Delta t} = b_4 \left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t \right) + b_5 \dot{\boldsymbol{U}}_t + b_6 \ddot{\boldsymbol{U}}_t \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t} \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_{t+\Delta t} \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\Delta t} \leftarrow \boldsymbol{F}\left(t + \Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots \right)$

Go to step II.1 with $t + \Delta t$.

A.6.2 Average Acceleration Method

The average acceleration method is identical to the trapezoidal rule, (see Figure A.8), in which we take the acceleration approach as follows:

$$\ddot{\boldsymbol{U}}_{t+\tau}(\tau) = \frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2} \quad ; \quad 0 \le \tau \le \Delta t \tag{A.71}$$

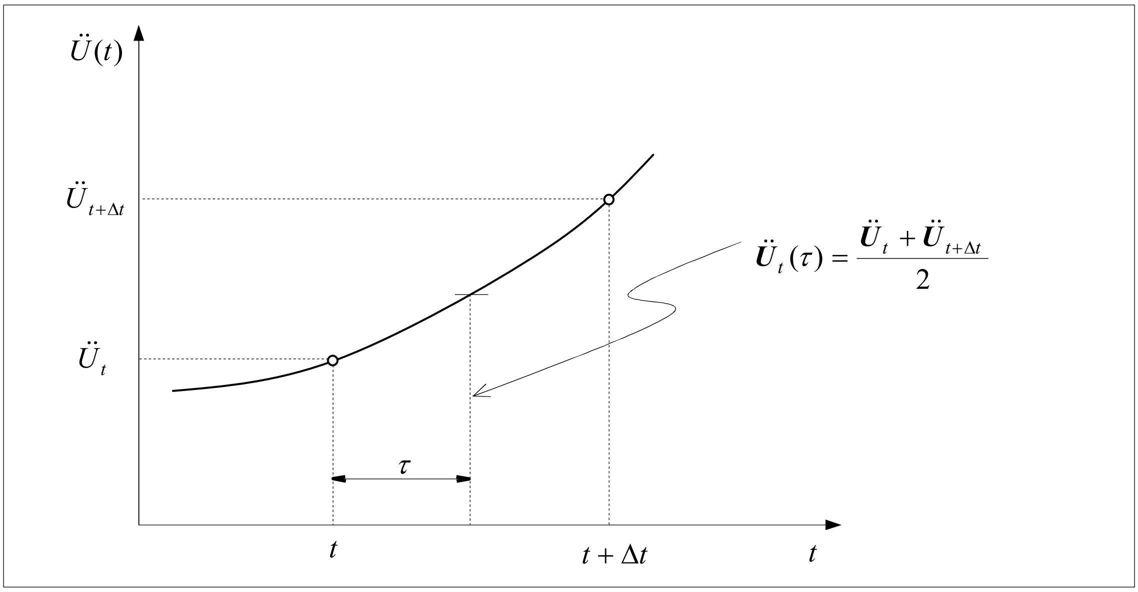

Figure A.8: Average acceleration.

Integrating the equation (A.71) we can obtain:

$$\dot{\boldsymbol{U}}_{t+\tau}(\tau) = \tau\left(\frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2}\right) + C_1 = \dot{\boldsymbol{U}}_t + \tau\left(\frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2}\right) \tag{A.72}$$

where the constant of integration C_1 was obtained by means of the initial condition $\dot{\boldsymbol{U}}_{t+\tau}(\tau = 0) = \dot{\boldsymbol{U}}_t \Rightarrow C_1 = \dot{\boldsymbol{U}}_t$. The displacement can be obtained by means of integration of the equation (A.72) over time, then:

$$\boldsymbol{U}_{t+\tau}(\tau) = \tau\dot{\boldsymbol{U}}_t + \frac{\tau^2}{2}\left(\frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2}\right) + C_2 = \boldsymbol{U}_t + \tau\dot{\boldsymbol{U}}_t + \frac{\tau^2}{2}\left(\frac{\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}}{2}\right) \tag{A.73}$$

Once again we use the initial condition to obtain the constant of integration $\boldsymbol{U}_{t+\tau}(\tau = 0) = \boldsymbol{U}_t \Rightarrow C_1 = \boldsymbol{U}_t$.

For $\tau = \Delta t$, the displacement and velocity vectors become:

$$\begin{aligned} \boldsymbol{U}_{t+\Delta t} &= \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{4}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{4}\ddot{\boldsymbol{U}}_{t+\Delta t} \\ \dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_{t+\Delta t} \end{aligned} \tag{A.74}$$

The equations given by (A.74) are the same equations given by the Newmark's Method when $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$, (see equations (A.49) and (A.50)).

By means of the equations (A.74) we can obtain the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\Delta t}$ in terms of $\boldsymbol{U}_{t+\Delta t}$ and $\boldsymbol{U}_t$, $\dot{\boldsymbol{U}}_t$, $\ddot{\boldsymbol{U}}_t$. By means of displacement vector given by (A.74) it is possible to obtain $\ddot{\boldsymbol{U}}_{t+\Delta t}$ as follows:

$$\ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{4}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{4}{\Delta t}\dot{\boldsymbol{U}}_t - \ddot{\boldsymbol{U}}_t \tag{A.75}$$

In turn we substitute the equation (A.75) into the velocity given by (A.74), thus:

$$\begin{aligned}\dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left[\frac{4}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{4}{\Delta t}\dot{\boldsymbol{U}}_t - \ddot{\boldsymbol{U}}_t\right] \\ &= \frac{2}{\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dot{\boldsymbol{U}}_t\end{aligned} \tag{A.76}$$

By substituting the velocity vector (A.76) and the acceleration vector (A.75) into the dynamic equilibrium equation at time $t+\Delta t$ we can obtain:

$$\begin{aligned}&\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} \\ &\boldsymbol{M}\left\{\frac{4}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{4}{\Delta t}\dot{\boldsymbol{U}}_t - \ddot{\boldsymbol{U}}_t\right\} + \boldsymbol{D}\left\{\frac{2}{\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dot{\boldsymbol{U}}_t\right\} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}\end{aligned} \tag{A.77}$$

By restructuring the above equation we can obtain

$$\left[\frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D} + \boldsymbol{K}\right]\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t} + \left[\frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{4}{\Delta t}\boldsymbol{M} + \boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \boldsymbol{M}\ddot{\boldsymbol{U}}_t \tag{A.78}$$

or

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}_{t+\Delta t} \tag{A.79}$$

where we have considered that:

$$\begin{aligned}\boldsymbol{K}^{eff} &= \frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D} + \boldsymbol{K} \\ \boldsymbol{F}^{eff}_{t+\Delta t} &= \boldsymbol{F}_{t+\Delta t} + \left[\frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{4}{\Delta t}\boldsymbol{M} + \boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \boldsymbol{M}\ddot{\boldsymbol{U}}_t\end{aligned} \tag{A.80}$$

A.6.2.1 Average Acceleration Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$ and $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Construction of $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D} + \boldsymbol{K}$$

I.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_0 \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_0$$

II. For each time step $t + \Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_{t+\Delta t} + \left[\frac{4}{\Delta t^2}\boldsymbol{M} + \frac{2}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{4}{\Delta t}\boldsymbol{M} + \boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \boldsymbol{M}\ddot{\boldsymbol{U}}_t$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$, $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\begin{cases} \dot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{2}{\Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dot{\boldsymbol{U}}_t \\ \ddot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{2}{\Delta t}\left[\dot{\boldsymbol{U}}_{t+\Delta t} - \dot{\boldsymbol{U}}_t\right] - \ddot{\boldsymbol{U}}_t \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t} \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_{t+\Delta t} \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\Delta t} \leftarrow \boldsymbol{F}\left(t + \Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots\right)$

Go to step II.1 with $t + \Delta t$.

A.6.3 Linear Acceleration Method

For this method we consider a linear variation for the acceleration field within the range $[t, t+\Delta t]$, (see Figure A.9):

$$\ddot{\boldsymbol{U}}_{t+\tau}(\tau) = \ddot{\boldsymbol{U}}_t + \frac{\tau}{\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.81}$$

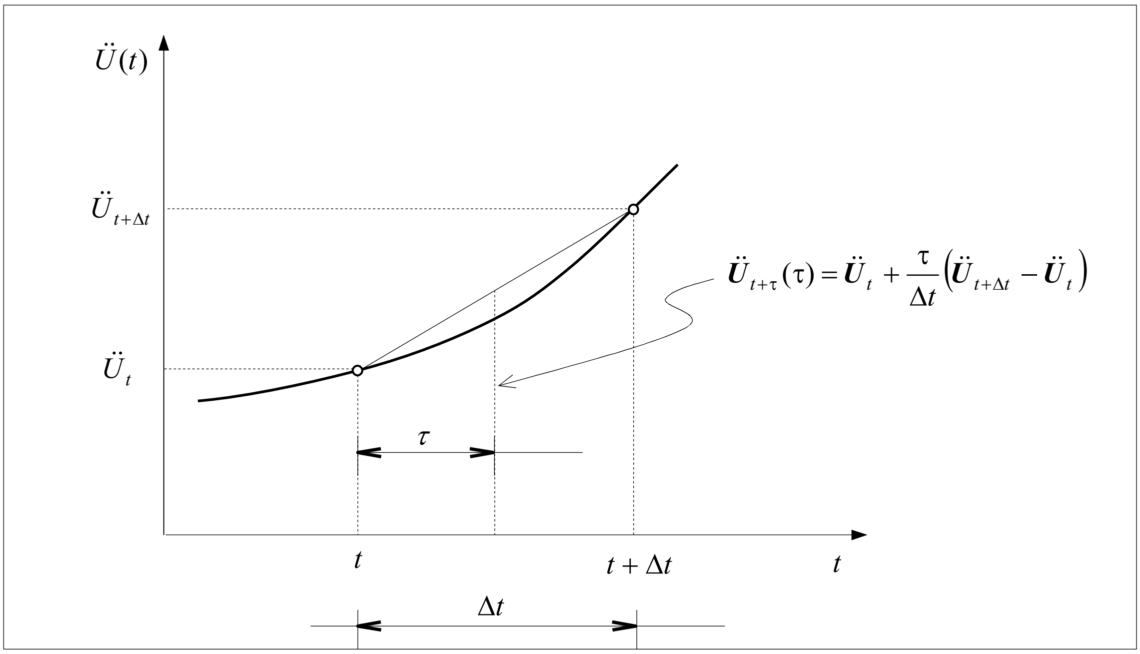

Figure A.9: Linear acceleration method.

The velocity vector can be obtained by integrate over time the equation (A.81), thus:

$$\begin{aligned}\dot{\boldsymbol{U}}_{t+\tau}(\tau) &= \tau\ddot{\boldsymbol{U}}_t + \frac{\tau^2}{2\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) + C_1 \\ &= \dot{\boldsymbol{U}}_t + \tau\ddot{\boldsymbol{U}}_t + \frac{\tau^2}{2\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right)\end{aligned} \tag{A.82}$$

where we have apply the initial condition in order to obtain the constant of integration, i.e. at $\tau = 0$ we have that $\dot{\boldsymbol{U}}_{t+\tau}(\tau = 0) = \dot{\boldsymbol{U}}_t \Rightarrow C_1 = \dot{\boldsymbol{U}}_t$.

Then, integrate over time the equation (A.82) we can obtain the displacement vector:

$$\begin{aligned}\boldsymbol{U}_{t+\tau}(\tau) &= \tau\dot{\boldsymbol{U}}_t + \frac{\tau^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\tau^3}{6\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) + C_2 \quad \therefore \quad \boldsymbol{U}_{t+\tau}(\tau = 0) = \boldsymbol{U}_t \Rightarrow C_2 = \boldsymbol{U}_t \\ &= \boldsymbol{U}_t + \tau\dot{\boldsymbol{U}}_t + \frac{\tau^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\tau^3}{6\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right)\end{aligned} \tag{A.83}$$

When $\tau = \Delta t$ we can obtain:

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.84}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \Delta t\ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) = \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + \ddot{\boldsymbol{U}}_t\right) \tag{A.85}$$

By means of the equation (A.84) it is possible to solve for $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{6}{\Delta t^2}(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t) - \frac{6}{\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t \tag{A.86}$$

In turn, by substituting (A.86) into (A.85) we can obtain:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{3}{\Delta t}(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t) - 2\dot{\boldsymbol{U}}_t - \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_t \tag{A.87}$$

Then, by substituting the velocity vector (A.87) and the acceleration vector (A.86) into the dynamic equilibrium equation we can obtain:

$$\begin{aligned}
&\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}\\
&\boldsymbol{M}\left\{\frac{6}{\Delta t^2}(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t) - \frac{6}{\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t\right\}\\
&\qquad + \boldsymbol{D}\left\{\frac{3}{\Delta t}(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t) - 2\dot{\boldsymbol{U}}_t - \frac{\Delta t}{2}\ddot{\boldsymbol{U}}_t\right\} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}
\end{aligned} \tag{A.88}$$

The above equation can be restructured as follows:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}_{t+\Delta t} \tag{A.89}$$

where

$$\begin{aligned}
\boldsymbol{K}^{eff} &= \frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D} + \boldsymbol{K}\\
\boldsymbol{F}^{eff}_{t+\Delta t} &= \boldsymbol{F}_{t+\Delta t} + \left[\frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{6}{\Delta t}\boldsymbol{M} + 2\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \left[2\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t
\end{aligned} \tag{A.90}$$

The linear acceleration method is the same as Newmark's method when $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$. This fact can be verified by substituting $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$ into the equations in (A.51), with which the displacement and velocity vectors become:

$$\begin{aligned}
\boldsymbol{U}_{t+\Delta t} &= \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \left(\frac{1}{2} - \frac{1}{6}\right)\Delta t^2\ddot{\boldsymbol{U}}_t + \frac{1}{6}\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{3}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6}\ddot{\boldsymbol{U}}_{t+\Delta t}\\
\dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + \left(1 - \frac{1}{2}\right)\Delta t\ddot{\boldsymbol{U}}_t + \frac{1}{2}\Delta t\;\ddot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left[\ddot{\boldsymbol{U}}_t + \ddot{\boldsymbol{U}}_{t+\Delta t}\right]
\end{aligned} \tag{A.91}$$

which match the equations obtained by using the linear acceleration method, (see equations (A.84) and (A.85)).

A.6.3.1 Linear Acceleration Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$ and $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Construction of $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D} + \boldsymbol{K}$$

I.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_0 \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_0$$

II. For each time step $t + \Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_{t+\Delta t} + \left[\frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t + \left[\frac{6}{\Delta t}\boldsymbol{M} + 2\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \left[2\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$, $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\begin{cases} \ddot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{6}{\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \dfrac{6}{\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t \\ \dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \dfrac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + \ddot{\boldsymbol{U}}_t\right) \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t} \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_{t+\Delta t} \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\Delta t} \leftarrow \boldsymbol{F}\left(t + \Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots\right)$

Go to step II.1 with $t + \Delta t$.

A.6.4 Central Finite Difference Method

This explicit method assumes that the velocity and acceleration vectors are obtained by using th central finite difference to approach the first and second derivatives:

$$\begin{aligned}\dot{\boldsymbol{U}}_t &= \frac{\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_{t-\Delta t}}{2\Delta t}\\ \ddot{\boldsymbol{U}}_t &= \frac{\boldsymbol{U}_{t+\Delta t} - 2\boldsymbol{U}_t + \boldsymbol{U}_{t-\Delta t}}{\Delta t^2}\end{aligned} \tag{A.92}$$

By substituting the equations in (A.92) into the dynamic equilibrium equation at time t, $\boldsymbol{M}\ddot{\boldsymbol{U}}_t + \boldsymbol{D}\dot{\boldsymbol{U}}_t + \boldsymbol{K}\boldsymbol{U}_t = \boldsymbol{F}_t$, we can obtain:

$$\begin{aligned}&\boldsymbol{M}\frac{\boldsymbol{U}_{t+\Delta t} - 2\boldsymbol{U}_t + \boldsymbol{U}_{t-\Delta t}}{\Delta t^2} + \boldsymbol{D}\frac{\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_{t-\Delta t}}{2\Delta t} + \boldsymbol{K}\boldsymbol{U}_t = \boldsymbol{F}_t\\ &\Rightarrow \left[\frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{2\Delta t}\right]\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_t + \left(\frac{2\boldsymbol{M}}{\Delta t^2} - \boldsymbol{K}\right)\boldsymbol{U}_t + \left(\frac{\boldsymbol{D}}{2\Delta t} - \frac{\boldsymbol{M}}{\Delta t^2}\right)\boldsymbol{U}_{t-\Delta t}\\ &\Rightarrow \boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}\end{aligned} \tag{A.93}$$

where

$$\begin{aligned}\boldsymbol{K}^{eff} &= \frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{2\Delta t}\\ \boldsymbol{F}^{eff} &= \boldsymbol{F}_t + \left(\frac{2\boldsymbol{M}}{\Delta t^2} - \boldsymbol{K}\right)\boldsymbol{U}_t + \left(\frac{\boldsymbol{D}}{2\Delta t} - \frac{\boldsymbol{M}}{\Delta t^2}\right)\boldsymbol{U}_{t-\Delta t}\end{aligned} \tag{A.94}$$

At time $t = 0$ we have to calculate $\boldsymbol{U}_{0-\Delta t}$, which value can be obtained by means of the equation (A.92):

$$\begin{aligned}\dot{\boldsymbol{U}}_0 &= \frac{\boldsymbol{U}_{0+\Delta t} - \boldsymbol{U}_{0-\Delta t}}{2\Delta t} \quad \Rightarrow \quad \boldsymbol{U}_{0+\Delta t} = 2\Delta t\dot{\boldsymbol{U}}_0 + \boldsymbol{U}_{0-\Delta t}\\ \ddot{\boldsymbol{U}}_0 &= \frac{\boldsymbol{U}_{0+\Delta t} - 2\boldsymbol{U}_0 + \boldsymbol{U}_{0-\Delta t}}{\Delta t^2} \quad \Rightarrow \quad \boldsymbol{U}_{0+\Delta t} = \Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - \boldsymbol{U}_{0-\Delta t}\end{aligned} \tag{A.95}$$

From the two above equations in (A.95) we can obtain:

$$\begin{aligned}&\Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - \boldsymbol{U}_{0-\Delta t} = 2\Delta t\dot{\boldsymbol{U}}_0 + \boldsymbol{U}_{0-\Delta t}\\ &\Rightarrow 2\boldsymbol{U}_{0-\Delta t} = \Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - 2\Delta t\dot{\boldsymbol{U}}_0\\ &\Rightarrow \boldsymbol{U}_{0-\Delta t} = \boldsymbol{U}_0 - \Delta t\dot{\boldsymbol{U}}_0 + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_0\end{aligned} \tag{A.96}$$

A.6.4.1 Central Finite Difference Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$ and $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Construction of $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{2\Delta t}$$

I.4. Calculate $\boldsymbol{U}_{0-\Delta t}$:

$$\boldsymbol{U}_{0-\Delta t} = \boldsymbol{U}_0 - \Delta t\dot{\boldsymbol{U}}_0 + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_0$$

I.4. Update the variables:

$$\boldsymbol{U}_{t-\Delta t} \leftarrow \boldsymbol{U}_{0-\Delta t} \quad ; \quad \boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad F_t = F_0$$

II. For each time step t do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_t + \left(\frac{2\boldsymbol{M}}{\Delta t^2} - \boldsymbol{K}\right)\boldsymbol{U}_t + \left(\frac{\boldsymbol{D}}{2\Delta t} - \frac{\boldsymbol{M}}{\Delta t^2}\right)\boldsymbol{U}_{t-\Delta t}$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_t$, $\ddot{\boldsymbol{U}}_t$:

$$\begin{cases} \dot{\boldsymbol{U}}_t = \dfrac{\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_{t-\Delta t}}{2\Delta t} \\ \ddot{\boldsymbol{U}}_t = \dfrac{\boldsymbol{U}_{t+\Delta t} - 2\boldsymbol{U}_t + \boldsymbol{U}_{t-\Delta t}}{\Delta t^2} \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_{t-\Delta t} \leftarrow \boldsymbol{U}_t \quad ; \quad \boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_t \leftarrow \boldsymbol{F}\left(t+\Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots\right)$

Go to step II.1 with $t + \Delta t$.

The central finite difference is a method with explicit integration and conditionally stable and requires time step Δt less than the critical value:

$$\Delta t \leq \Delta t_{cr} = \frac{T_{\min}}{\pi} = \frac{2}{\omega_{\max}} \tag{A.97}$$

where $T_{\min}$ stands for the smallest natural period, and $\omega_{\max}$ is the maximum frequency of the discrete system, which is greater eigenvalue of the characteristic determinant:

$$\det(\boldsymbol{K} + \omega^2 \boldsymbol{M}) \equiv \left|\boldsymbol{K} + \omega^2 \boldsymbol{M}\right| = \left|\boldsymbol{K} + \lambda \boldsymbol{M}\right| = 0 \tag{A.98}$$

A.6.5 Wilson-$\boldsymbol{\theta}$ Method

The Wilson-θ method (1968) is an extension of the linear acceleration method (Newmark's method when $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$). The acceleration vector is approached within the range $0 \leq \tau \leq \theta \Delta t$ as showed in Figure A.10, where $\theta \geq 1$. When $\theta = 1$ we fall back into the linear acceleration method.

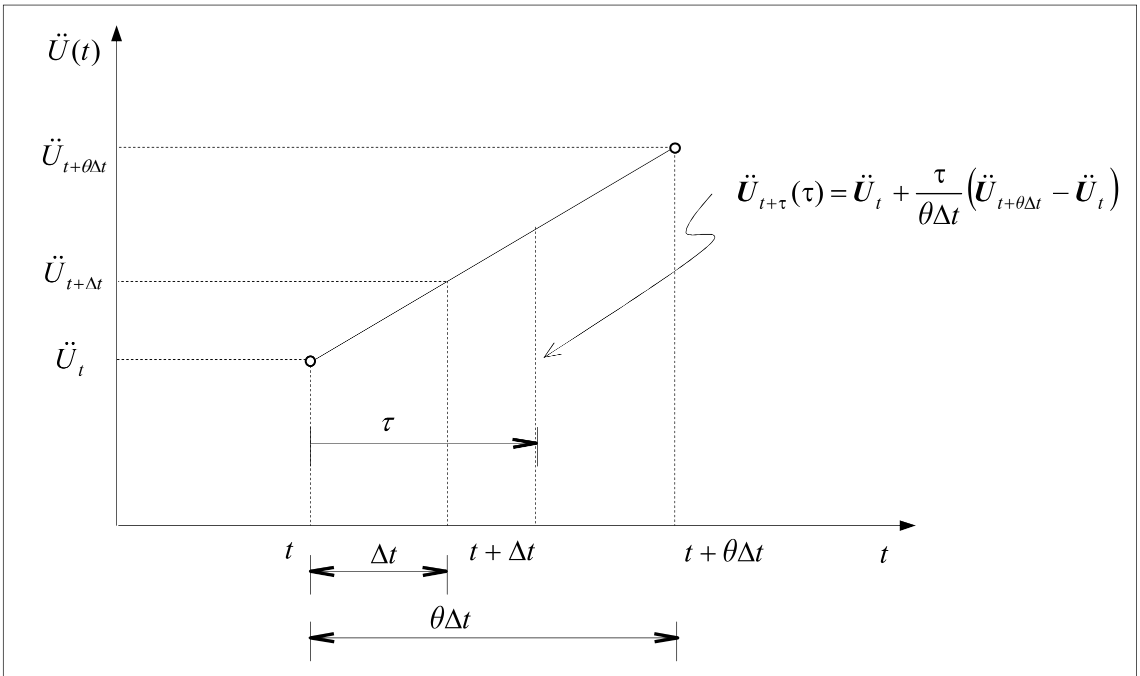

Figure A.10: Acceleration approach – Wilson-θ method.

By means of Figure A.10 the acceleration vector becomes:

$$\ddot{\boldsymbol{U}}_{t+\tau}(\tau) = \ddot{\boldsymbol{U}}_t + \frac{\tau}{\theta \Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.99}$$

By integrate over time the above equation we can obtain the velocity vector:

$$\dot{\boldsymbol{U}}_{t+\tau}(\tau) = \dot{\boldsymbol{U}}_t + \tau \ddot{\boldsymbol{U}}_t + \frac{\tau^2}{2\theta \Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.100}$$

In turn, by integrate the above equation (A.100) we can obtain the displacement vector:

$$\boldsymbol{U}_{t+\tau}(\tau) = \boldsymbol{U}_t + \tau \dot{\boldsymbol{U}}_t + \frac{\tau^2}{2} \ddot{\boldsymbol{U}}_t + \frac{\tau^3}{6\theta \Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.101}$$

By considering $\tau = \theta \Delta t$ the velocity vector becomes:

$$\begin{aligned}\dot{\boldsymbol{U}}_{t+\theta\Delta t} &= \dot{\boldsymbol{U}}_t + \theta\Delta t\ddot{\boldsymbol{U}}_t + \frac{(\theta\Delta t)^2}{2\theta\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right)\\ &= \dot{\boldsymbol{U}}_t + \frac{\theta\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} + \ddot{\boldsymbol{U}}_t\right)\end{aligned} \tag{A.102}$$

and the displacement vector:

$$\begin{aligned}\boldsymbol{U}_{t+\theta\Delta} &= \boldsymbol{U}_t + \theta\Delta t\dot{\boldsymbol{U}}_t + \frac{(\theta\Delta t)^2}{2}\ddot{\boldsymbol{U}}_t + \frac{(\theta\Delta t)^3}{6\theta\Delta t}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right)\\ &= \boldsymbol{U}_t + \theta\Delta t\dot{\boldsymbol{U}}_t + \frac{\theta^2\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} + 2\ddot{\boldsymbol{U}}_t\right)\end{aligned} \tag{A.103}$$

Then, by solving for $\ddot{\boldsymbol{U}}_{t+\theta\Delta t}$:

$$\ddot{\boldsymbol{U}}_{t+\theta\Delta t} = \frac{6}{\theta^2\Delta t^2}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - \frac{6}{\theta\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t \tag{A.104}$$

By substituting the equation (A.104) into the velocity equation (A.102) we can obtain:

$$\dot{\boldsymbol{U}}_{t+\theta\Delta t} = \frac{3}{\theta\Delta t}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - 2\dot{\boldsymbol{U}}_t - \frac{\theta\Delta t}{2}\ddot{\boldsymbol{U}}_t \tag{A.105}$$

Taking into account the dynamic equilibrium equation at time $t+\theta\Delta t$:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\theta\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\theta\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\theta\Delta t} = \hat{\boldsymbol{F}}_{t+\theta\Delta t} \tag{A.106}$$

where $\hat{\boldsymbol{F}}_{t+\theta\Delta t} = \theta\boldsymbol{F}_{t+\theta\Delta t} + (1-\theta)\boldsymbol{F}_t$, and by substituting the values for $\dot{\boldsymbol{U}}_{t+\theta\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\theta\Delta t}$ given respectively by the equations (A.105) and (A.104), we can obtain the following set of equations:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\theta\Delta t} = \boldsymbol{F}^{eff} \tag{A.107}$$

where

$$\begin{aligned}\boldsymbol{K}^{eff} &= \frac{6\boldsymbol{M}}{\theta^2\Delta t^2} + \frac{3\boldsymbol{D}}{\theta\Delta t} + \boldsymbol{K}\\ \boldsymbol{F}^{eff} &= \theta\boldsymbol{F}_{t+\theta\Delta t} + (1-\theta)\boldsymbol{F}_t + \left(\frac{6\boldsymbol{M}}{\theta^2\Delta t^2} + \frac{3\boldsymbol{D}}{\theta\Delta t}\right)\boldsymbol{U}_t + \left(\frac{6\boldsymbol{M}}{\theta\Delta t} + 2\boldsymbol{D}\right)\dot{\boldsymbol{U}}_t + \left(2\boldsymbol{M} + \frac{\theta\Delta t}{2}\boldsymbol{D}\right)\ddot{\boldsymbol{U}}_t\end{aligned} \tag{A.108}$$

After the system (A.107) is solved, $\boldsymbol{U}_{t+\theta\Delta t}$ is determined and is possible to calculate $\boldsymbol{U}_{t+\Delta t}$, $\dot{\boldsymbol{U}}_{t+\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\Delta t}$. To do this, we consider the equation (A.99) when $\tau = \Delta t$, thus:

$$\ddot{\boldsymbol{U}}_{t+\Delta t} = \ddot{\boldsymbol{U}}_t + \frac{1}{\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.109}$$

By substituting the value of $\ddot{\boldsymbol{U}}_{t+\theta\Delta t}$ given by the equation (A.104), the above equation becomes:

$$\begin{aligned}\ddot{\boldsymbol{U}}_{t+\Delta t} &= \ddot{\boldsymbol{U}}_t + \frac{1}{\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right)\\ &= \ddot{\boldsymbol{U}}_t + \frac{1}{\theta}\left[\frac{6}{\theta^2\Delta t^2}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - \frac{6}{\theta\Delta t}\dot{\boldsymbol{U}}_t - 2\ddot{\boldsymbol{U}}_t - \ddot{\boldsymbol{U}}_t\right]\\ &= \frac{6}{\theta^3\Delta t^2}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - \frac{6}{\theta^2\Delta t}\dot{\boldsymbol{U}}_t + \left(1 - \frac{3}{\theta}\right)\ddot{\boldsymbol{U}}_t\end{aligned} \tag{A.110}$$

To obtain $\dot{\boldsymbol{U}}_{t+\Delta t}$ we use the equation in (A.100) by assuming $\tau = \Delta t$, thus:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \Delta t \ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \tag{A.111}$$

Taking into account the equation (A.109), the relationship $\frac{1}{\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) = \ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t$ holds, and substituting into the above equation we can obtain:

$$\begin{aligned} \dot{\boldsymbol{U}}_{t+\Delta t} &= \dot{\boldsymbol{U}}_t + \Delta t \ddot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) \\ &= \dot{\boldsymbol{U}}_t + \frac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + \ddot{\boldsymbol{U}}_t\right) \end{aligned} \tag{A.112}$$

In order to obtain the displacement $\boldsymbol{U}_{t+\Delta t}$ it is enough to enforce $\tau = \Delta t$ in the equation (A.101), thus:

$$\begin{aligned} \boldsymbol{U}_{t+\Delta t} &= \boldsymbol{U}_t + \Delta t \dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6\theta}\left(\ddot{\boldsymbol{U}}_{t+\theta\Delta t} - \ddot{\boldsymbol{U}}_t\right) \\ &= \boldsymbol{U}_t + \Delta t \dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{2}\ddot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} - \ddot{\boldsymbol{U}}_t\right) \\ &= \boldsymbol{U}_t + \Delta t \dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + 2\ddot{\boldsymbol{U}}_t\right) \end{aligned} \tag{A.113}$$

A.6.5.1 Wilson-θ Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$, $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Calculate $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{6\boldsymbol{M}}{\theta^2\Delta t^2} + \frac{3\boldsymbol{D}}{\theta\Delta t} + \boldsymbol{K}$$

I.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_0 \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_0 \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_0$$

II. For each time step $t+\Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \theta\boldsymbol{F}_{t+\theta\Delta t} + (1-\theta)\boldsymbol{F}_t + \left(\frac{6\boldsymbol{M}}{\theta^2\Delta t^2} + \frac{3\boldsymbol{D}}{\theta\Delta t}\right)\boldsymbol{U}_t + \left(\frac{6\boldsymbol{M}}{\theta\Delta t} + 2\boldsymbol{D}\right)\dot{\boldsymbol{U}}_t + \left(2\boldsymbol{M} + \frac{\theta\Delta t}{2}\boldsymbol{D}\right)\ddot{\boldsymbol{U}}_t$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\theta\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\ddot{\boldsymbol{U}}_{t+\Delta t}$, $\dot{\boldsymbol{U}}_{t+\Delta t}$ and $\boldsymbol{U}_{t+\Delta t}$:

$$\begin{cases} \ddot{\boldsymbol{U}}_{t+\Delta t} = \dfrac{6}{\theta^3\Delta t^2}\left(\boldsymbol{U}_{t+\theta\Delta} - \boldsymbol{U}_t\right) - \dfrac{6}{\theta^2\Delta t}\dot{\boldsymbol{U}}_t + \left(1 - \dfrac{3}{\theta}\right)\ddot{\boldsymbol{U}}_t \\ \dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + \dfrac{\Delta t}{2}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + \ddot{\boldsymbol{U}}_t\right) \\ \boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \dfrac{\Delta t^2}{6}\left(\ddot{\boldsymbol{U}}_{t+\Delta t} + 2\ddot{\boldsymbol{U}}_t\right) \end{cases}$$

II.4. Update the variables:

$$\boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t} \quad ; \quad \dot{\boldsymbol{U}}_t \leftarrow \dot{\boldsymbol{U}}_{t+\Delta t} \quad ; \quad \ddot{\boldsymbol{U}}_t \leftarrow \ddot{\boldsymbol{U}}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\theta\Delta t} \leftarrow \boldsymbol{F}\left(t+\theta\Delta t, \boldsymbol{U}_{t+\theta\Delta t}, \dot{\boldsymbol{U}}_{t+\theta\Delta t}, \ddot{\boldsymbol{U}}_{t+\theta\Delta t}, \ldots\right)$

Go to step II.1 with $t+\Delta t$.

A.6.6 Houbolt's Method

Houbolt's method (1950) is a mixed method, and approaches the displacement by a cubic function within the range between $t-2\Delta t$ and $t+\Delta t$, according to the function

$$U_{t+\tau}(\tau) = a\tau^3 + b\tau^2 + c\tau + d \tag{A.114}$$

where the coefficients a, b, c and d are given by:

$$\begin{aligned} a &= \frac{1}{6\Delta t^3}\left[-U_{t-2\Delta t} + 3U_{t-\Delta t} - 3U_t + U_{t+\Delta t}\right] \quad ; \quad b = \frac{1}{2\Delta t^2}\left[U_{t-\Delta t} - 2U_t + U_{t+\Delta t}\right] \\ c &= \frac{1}{6\Delta t}\left[U_{t-2\Delta t} - 6U_{t-\Delta t} + 3U_t + 2U_{t+\Delta t}\right] \quad ; \quad d = U_t \end{aligned} \tag{A.115}$$

The first and second derivatives in (A.114) are given by:

$$\dot{U}_{t+\tau}(\tau) = 3a\tau^2 + 2b\tau + c \quad ; \quad \ddot{U}_{t+\tau}(\tau) = 6a\tau + 2b \tag{A.116}$$

Assuming $\tau = \Delta t$ we can obtain:

$$\dot{U}_{t+\Delta t} = 3a\Delta t^2 + 2b\Delta t + c \quad ; \quad \ddot{U}_{t+\Delta t} = 6a\Delta t + 2b \tag{A.117}$$

By substituting the values for a, b and c given by (A.115) into the equations in (A.117) we can obtain:

$$\begin{aligned} \dot{U}_{t+\Delta t} &= \frac{1}{6\Delta t}\left[11U_{t+\Delta t} - 18U_t + 9U_{t-\Delta t} - 2U_{t-2\Delta t}\right] \\ \ddot{U}_{t+\Delta t} &= \frac{1}{\Delta t^2}\left[2U_{t+\Delta t} - 5U_t + 4U_{t-\Delta t} - U_{t-2\Delta t}\right] \end{aligned} \tag{A.118}$$

NOTE: This method is unconditionally stable but provides artificial damping (numerical damping) which is very high for a low-frequency response.

By considering the dynamic equation at $t+\Delta t$, i.e. $\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} + \boldsymbol{K}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}_{t+\Delta t}$, and by substituting the values for $\dot{\boldsymbol{U}}_{t+\Delta t}$ and $\ddot{\boldsymbol{U}}_{t+\Delta t}$ given by (A.118), we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff} \tag{A.119}$$

where

$$\begin{aligned} \boldsymbol{K}^{eff} &= \frac{2}{\Delta t^2}\boldsymbol{M} + \frac{11}{6\Delta t}\boldsymbol{D} + \boldsymbol{K} \\ \boldsymbol{F}^{eff} &= \boldsymbol{F}_{t+\Delta t} + \left(\frac{5\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_t - \left(\frac{4\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{2\Delta t}\right)\boldsymbol{U}_{t-\Delta t} + \left(\frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{3\Delta t}\right)\boldsymbol{U}_{t-2\Delta t} \end{aligned} \tag{A.120}$$

Similarly to the central finite difference, the Houbolt's method needs a pretreatment at $t=0$ in order to obtain the values for $U_{0-\Delta t}$ and $U_{0-2\Delta t}$. Then, we express the values for $\dot{U}_{t+\tau}(\tau)$ and $\ddot{U}_{t+\tau}(\tau)$, (see equations in (A.116)), at time $\tau = 0$, thus

$$\begin{aligned} \dot{U}_{t+0}(\tau=0) &= c = \frac{1}{6\Delta t}\left[U_{t-2\Delta t} - 6U_{t-\Delta t} + 3U_t + 2U_{t+\Delta t}\right] \\ \ddot{U}_{t+0}(\tau=0) &= 2b = \frac{1}{\Delta t^2}\left[U_{t-\Delta t} - 2U_t + U_{t+\Delta t}\right] \end{aligned} \tag{A.121}$$

That is, we apply in the third point of integration, (see Figure A.11).

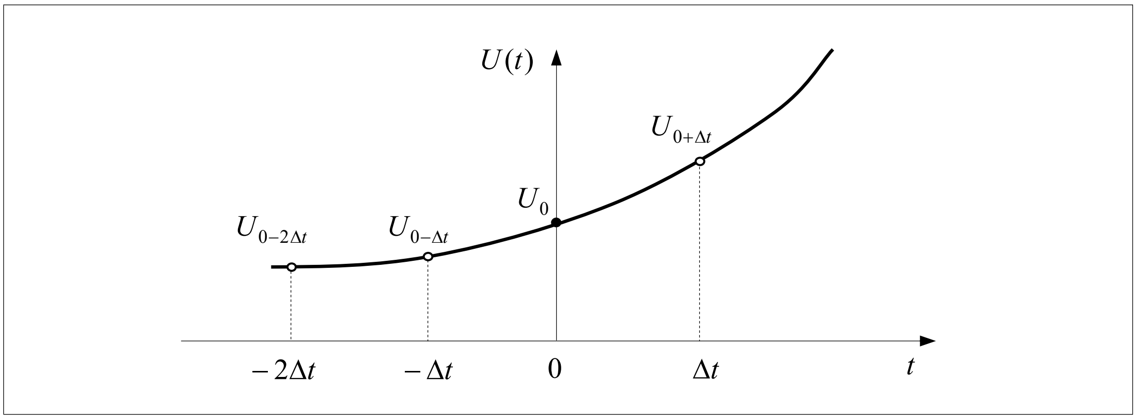

Figure A.11: Houbolt's method parameters at $t=0$.

Assuming the equations (A.121), at time $t=0$, we can obtain:

$$\dot{U}_0 = \frac{1}{6\Delta t}\left[U_{0-2\Delta t} - 6U_{0-\Delta t} + 3U_0 + 2U_{0+\Delta t}\right] \quad ; \quad \ddot{U}_0 = \frac{1}{\Delta t^2}\left[U_{0-\Delta t} - 2U_0 + U_{0+\Delta t}\right] \tag{A.122}$$

The value for $U_{0-\Delta t}$ can be obtained by means of the acceleration equation given by (A.122), i.e.:

$$U_{0-\Delta t} = \Delta t^2 \ddot{U}_0 + 2U_0 - U_{0+\Delta t} \tag{A.123}$$

Then, by substituting the value $U_{0-\Delta t}$ into the velocity equation ($\dot{U}_0$), given by (A.122), it is possible to obtain the value for $U_{0-2\Delta t}$, i.e.:

$$U_{0-2\Delta t} = 6\Delta t \dot{U}_0 + 6\Delta t^2 \ddot{U}_0 - 8U_{0+\Delta t} + 9U_0 \tag{A.124}$$

At time $t=0$ the set of equations (A.119) becomes:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{0+\Delta t} = \boldsymbol{F}^{eff} \tag{A.125}$$

where

$$\begin{aligned}
\boldsymbol{K}^{eff} &= \frac{2}{\Delta t^2}\boldsymbol{M} + \frac{11}{6\Delta t}\boldsymbol{D} + \boldsymbol{K} \\
\boldsymbol{F}^{eff} &= \boldsymbol{F}_{0+\Delta t} + \left(\frac{5\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_0 - \left(\frac{4\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{2\Delta t}\right)\boldsymbol{U}_{0-\Delta t} + \left(\frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{3\Delta t}\right)\boldsymbol{U}_{0-2\Delta t}
\end{aligned} \tag{A.126}$$

Note that $\boldsymbol{U}_{0-\Delta t}$ and $\boldsymbol{U}_{0-2\Delta t}$ are also functions of the unknown $\boldsymbol{U}_{0+\Delta t}$. By substituting the values for $\boldsymbol{U}_{0-\Delta t}$ and $\boldsymbol{U}_{0-2\Delta t}$ given by (A.123) and (A.124) into the set of equations (A.125) and after restructuring we can obtain:

$$\hat{\boldsymbol{K}}^{eff}\boldsymbol{U}_{0+\Delta t} = \hat{\boldsymbol{F}}^{eff} \tag{A.127}$$

$$\begin{aligned}
\hat{\boldsymbol{K}}^{eff} &= \frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D} + \boldsymbol{K} \\
\hat{\boldsymbol{F}}^{eff} &= \boldsymbol{F}_{0+\Delta t} + \left(\frac{6\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_0 + \left(\frac{6\boldsymbol{M}}{\Delta t} + 2\boldsymbol{D}\right)\dot{\boldsymbol{U}}_0 + \left(2\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}\right)\ddot{\boldsymbol{U}}_0
\end{aligned} \tag{A.128}$$

The value for $\boldsymbol{U}_{0+\Delta t}$ is obtained after the system (A.127) is solved, with which the values for $\boldsymbol{U}_{0-\Delta t}$ and $\boldsymbol{U}_{0-2\Delta t}$ can be obtained by means of the equations (A.123) and (A.124), respectively. Note that the equations in (A.128) are the same equations obtained by linear acceleration method, (see equations in (A.90)).

A.6.6.1 Houbolt's Method Scheme

I. Initial Parameters

I.1. Construction of $\boldsymbol{M}$, $\boldsymbol{D}$, $\boldsymbol{K}$.

I.2. Given the boundary conditions $\boldsymbol{U}_0$ and $\dot{\boldsymbol{U}}_0$, obtain $\ddot{\boldsymbol{U}}_0$:

$$\ddot{\boldsymbol{U}}_0 = \boldsymbol{M}^{-1}\left(\boldsymbol{F}_0 - \boldsymbol{D}\dot{\boldsymbol{U}}_0 - \boldsymbol{K}\boldsymbol{U}_0\right)$$

I.3. Calculate the matrices:

$$\hat{\boldsymbol{K}}^{eff} = \frac{6}{\Delta t^2}\boldsymbol{M} + \frac{3}{\Delta t}\boldsymbol{D} + \boldsymbol{K}$$

$$\hat{\boldsymbol{F}}^{eff} = \boldsymbol{F}_{0+\Delta t} + \left(\frac{6\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_0 + \left(\frac{6\boldsymbol{M}}{\Delta t} + 2\boldsymbol{D}\right)\dot{\boldsymbol{U}}_0 + \left(2\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}\right)\ddot{\boldsymbol{U}}_0$$

I.4. Solve the system:

$$\hat{\boldsymbol{K}}^{eff}\boldsymbol{U}_{0+\Delta t} = \hat{\boldsymbol{F}}^{eff}$$

I.5. Calculate the vectors $\boldsymbol{U}_{0-\Delta t}$, $\boldsymbol{U}_{0-2\Delta t}$ and $\dot{\boldsymbol{U}}_{0+\Delta t}$, $\ddot{\boldsymbol{U}}_{0+\Delta t}$:

$$\boldsymbol{U}_{0-\Delta t} = \Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - \boldsymbol{U}_{0+\Delta t} \text{ y } \boldsymbol{U}_{0-2\Delta t} = 6\Delta t\dot{\boldsymbol{U}}_0 + 6\Delta t^2\ddot{\boldsymbol{U}}_0 - 8\boldsymbol{U}_{0+\Delta t} + 9\boldsymbol{U}_0$$

$$\boldsymbol{U}_{0-\Delta t} = \Delta t^2\ddot{\boldsymbol{U}}_0 + 2\boldsymbol{U}_0 - \boldsymbol{U}_{0+\Delta t} \quad ; \quad \boldsymbol{U}_{0-2\Delta t} = 6\Delta t\dot{\boldsymbol{U}}_0 + 6\Delta t^2\ddot{\boldsymbol{U}}_0 - 8\boldsymbol{U}_{0+\Delta t} + 9\boldsymbol{U}_0$$

$$\dot{\boldsymbol{U}}_{0+\Delta t} = \frac{1}{6\Delta t}\left[11\boldsymbol{U}_{0+\Delta t} - 18\boldsymbol{U}_0 + 9\boldsymbol{U}_{0-\Delta t} - 2\boldsymbol{U}_{0-2\Delta t}\right] \quad ; \quad \ddot{\boldsymbol{U}}_{0+\Delta t} = \frac{1}{\Delta t^2}\left[2\boldsymbol{U}_{0+\Delta t} - 5\boldsymbol{U}_0 + 4\boldsymbol{U}_{0-\Delta t} - \boldsymbol{U}_{0-2\Delta t}\right]$$

I.6. Calculate the effective matrix $\boldsymbol{K}^{eff}$:

$$\boldsymbol{K}^{eff} = \frac{2}{\Delta t^2}\boldsymbol{M} + \frac{11}{6\Delta t}\boldsymbol{D} + \boldsymbol{K}$$

I.7. Update the variables:

$$\boldsymbol{U}_{t-2\Delta t} \leftarrow \boldsymbol{U}_{0-\Delta t} \quad ; \quad \boldsymbol{U}_{t-\Delta t} \leftarrow \boldsymbol{U}_0 \quad ; \quad \boldsymbol{U}_t \leftarrow \boldsymbol{U}_{0+\Delta t}$$

II. For each time step $t + \Delta t$ do

II.1. Obtain the effective force vector:

$$\boldsymbol{F}^{eff} = \boldsymbol{F}_{t+\Delta t} + \left(\frac{5\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{\Delta t}\right)\boldsymbol{U}_t - \left(\frac{4\boldsymbol{M}}{\Delta t^2} + \frac{3\boldsymbol{D}}{2\Delta t}\right)\boldsymbol{U}_{t-\Delta t} + \left(\frac{\boldsymbol{M}}{\Delta t^2} + \frac{\boldsymbol{D}}{3\Delta t}\right)\boldsymbol{U}_{t-2\Delta t}$$

II.2. Solve the system:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff}$$

II.3. Calculate the vectors $\dot{\boldsymbol{U}}_{t+\Delta t}$, $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{6\Delta t}\left[11\boldsymbol{U}_{t+\Delta t} - 18\boldsymbol{U}_t + 9\boldsymbol{U}_{t-\Delta t} - 2\boldsymbol{U}_{t-2\Delta t}\right] \quad ; \quad \ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\Delta t^2}\left[2\boldsymbol{U}_{t+\Delta t} - 5\boldsymbol{U}_t + 4\boldsymbol{U}_{t-\Delta t} - \boldsymbol{U}_{t-2\Delta t}\right]$$

II.4. Update the variables:

$$\boldsymbol{U}_{t-2\Delta t} \leftarrow \boldsymbol{U}_{t-\Delta t} \quad ; \quad \boldsymbol{U}_{t-\Delta t} \leftarrow \boldsymbol{U}_t \quad ; \quad \boldsymbol{U}_t \leftarrow \boldsymbol{U}_{t+\Delta t}$$

If it is the case $\boldsymbol{F}_{t+\Delta t} \leftarrow \boldsymbol{F}\left(t + \Delta t, \boldsymbol{U}_{t+\Delta t}, \dot{\boldsymbol{U}}_{t+\Delta t}, \ddot{\boldsymbol{U}}_{t+\Delta t}, \ldots\right)$

Go to step II.1 with $t + \Delta t$.

A.6.7 Hilber-Hughes-Taylor's Method (HHT)

The HHT method adopts the same displacement and velocity approximations used by Newmark's method. The difference between these two methods is how they treat the dynamic equilibrium equation:

$$\begin{aligned}
&\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t\dot{\boldsymbol{U}}_t + \left(\frac{1}{2} - \beta_H\right)\Delta t^2\ddot{\boldsymbol{U}}_t + \beta_H \Delta t^2 \ddot{\boldsymbol{U}}_{t+\Delta t} \\
&\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + (1-\gamma_H)\Delta t\ddot{\boldsymbol{U}}_t + \gamma_H \Delta t\ \ddot{\boldsymbol{U}}_{t+\Delta t} \\
&\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t} + (1+\alpha_H)\boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t} - \alpha_H \boldsymbol{D}\dot{\boldsymbol{U}}_t + (1+\alpha_H)\boldsymbol{K}\boldsymbol{U}_{t+\Delta t} - \alpha_H \boldsymbol{K}\boldsymbol{U}_t = \boldsymbol{F}_{t+\Delta t}
\end{aligned} \quad \text{Hilber-Hughes-Taylor's Method} \qquad (A.129)$$

with $\alpha_H < 0$.

By means of the displacement $\boldsymbol{U}_{t+\Delta t}$, given by the equation (A.129), we can obtain $\ddot{\boldsymbol{U}}_{t+\Delta t}$ as follows:

$$\ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\beta_H \Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{1}{\beta_H \Delta t}\dot{\boldsymbol{U}}_t - \left(\frac{1}{2\beta_H} - 1\right)\ddot{\boldsymbol{U}}_t \qquad (A.130)$$

By substituting the equation (A.130) into the velocity equation $\dot{\boldsymbol{U}}_{t+\Delta t}$, given by (A.129), we can obtain:

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{\gamma_H}{\beta_H \Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \left(1 - \frac{\gamma_H}{\beta_H}\right)\dot{\boldsymbol{U}}_t + \left(1 - \frac{\gamma_H}{2\beta_H}\right)\Delta t\ddot{\boldsymbol{U}}_t \qquad (A.131)$$

By substituting (A.130) and (A.131) into the dynamic equilibrium equation (A.129), we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff} \qquad (A.132)$$

where

$$\begin{aligned}
\boldsymbol{K}^{eff} &= \frac{1}{\beta_H \Delta t^2}\boldsymbol{M} + \frac{(1+\alpha_H)\gamma_H}{\beta_H \Delta t}\boldsymbol{D} + (1+\alpha_H)\boldsymbol{K} \\
\boldsymbol{F}^{eff} &= \boldsymbol{F}_{t+\Delta t} + \left[\frac{\boldsymbol{M}}{\beta_H \Delta t^2} + \frac{(1+\alpha_H)\gamma_H}{\beta_H \Delta t}\boldsymbol{D} + \alpha_H \boldsymbol{K}\right]\boldsymbol{U}_t + \\
&+ \left[\frac{\boldsymbol{M}}{\beta_H \Delta t} - \left(1 - \frac{\gamma_H(1+\alpha_H)}{\beta_H}\right)\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \left[\left(\frac{1}{2\beta_H} - 1\right)\boldsymbol{M} - (1+\alpha_H)\left(1 - \frac{\gamma_H}{2\beta_H}\right)\Delta t\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t
\end{aligned} \qquad (A.133)$$

For the particular case when $\alpha_H = 0$ we fall back into the Newmark's method. Besides, this method has second-order accuracy and is unconditionally stable when:

$$\frac{-1}{3} \le \alpha_H \le 0 \quad ; \quad \gamma_H = \frac{1}{2}(1-2\alpha_H) \quad ; \quad \beta_H = \frac{1}{4}(1-\alpha_H)^2 \qquad (A.134)$$

The smaller α_H greater the numerical damping is. The numerical damping is small for a low frequency response and will be high for high frequency response.

A.6.8 Bossak's Method

The Bossak's method uses the same displacement and velocity approaches used by Newmark's method. And for the Bossak's method the dynamic equilibrium equation is treated as follows:

$$\begin{array}{|l|}\hline \boldsymbol{U}_{t+\Delta t}=\boldsymbol{U}_t+\Delta t\dot{\boldsymbol{U}}_t+\left(\frac{1}{2}-\beta_B\right)\Delta t^2\ddot{\boldsymbol{U}}_t+\beta_B\Delta t^2\ddot{\boldsymbol{U}}_{t+\Delta t}\\ \dot{\boldsymbol{U}}_{t+\Delta t}=\dot{\boldsymbol{U}}_t+(1-\gamma_B)\Delta t\ddot{\boldsymbol{U}}_t+\gamma_B\Delta t\;\ddot{\boldsymbol{U}}_{t+\Delta t}\\ (1-\alpha_B)\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t}+\alpha_B\boldsymbol{M}\ddot{\boldsymbol{U}}_t+\boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t}+\boldsymbol{K}\boldsymbol{U}_{t+\Delta t}=\boldsymbol{F}_{t+\Delta t}\\ \hline\end{array}\quad \textit{Bossak's method} \tag{A.135}$$

When $\alpha_B=0$ we fall back into the Newmark's method.

The stability condition is met when:

$$\alpha_B\leq\frac{1}{2}\quad;\quad\beta_B\geq\frac{\gamma_B}{2}\geq\frac{1}{4}\quad;\quad\alpha_B+\beta_B\geq\frac{1}{2} \tag{A.136}$$

Using the displacement vector $\boldsymbol{U}_{t+\Delta t}$, given by (A.135), we can obtain the acceleration vector $\ddot{\boldsymbol{U}}_{t+\Delta t}$:

$$\ddot{\boldsymbol{U}}_{t+\Delta t}=\frac{1}{\beta_B\Delta t^2}\left(\boldsymbol{U}_{t+\Delta t}-\boldsymbol{U}_t\right)-\frac{1}{\beta_B\Delta t}\dot{\boldsymbol{U}}_t-\left(\frac{1}{2\beta_B}-1\right)\ddot{\boldsymbol{U}}_t \tag{A.137}$$

By substituting the equation (A.137) into the velocity equation $\dot{\boldsymbol{U}}_{t+\Delta t}$, given by (A.135), we can obtain:

$$\dot{\boldsymbol{U}}_{t+\Delta t}=\frac{\gamma_B}{\beta_B\Delta t}\left(\boldsymbol{U}_{t+\Delta t}-\boldsymbol{U}_t\right)+\left(1-\frac{\gamma_B}{\beta_B}\right)\dot{\boldsymbol{U}}_t+\left(1-\frac{\gamma_B}{2\beta_B}\right)\Delta t\ddot{\boldsymbol{U}}_t \tag{A.138}$$

By substituting (A.137) and (A.138) into the dynamic equilibrium equation (A.135), we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t}=\boldsymbol{F}^{eff} \tag{A.139}$$

where

$$\begin{aligned}\boldsymbol{K}^{eff}&=\frac{(1-\alpha_B)}{\beta_B\Delta t^2}\boldsymbol{M}+\frac{\gamma_B}{\beta_B\Delta t}\boldsymbol{D}+\boldsymbol{K}\\ \boldsymbol{F}^{eff}&=\boldsymbol{F}_{t+\Delta t}+\left[\frac{(1-\alpha_B)}{\beta_B\Delta t^2}\boldsymbol{M}+\frac{\gamma_B}{\beta_B\Delta t}\boldsymbol{D}\right]\boldsymbol{U}_t+\\ &\quad+\left[\frac{(1-\alpha_B)}{\beta_B\Delta t}\boldsymbol{M}-\left(1-\frac{\gamma_B}{\beta_B}\right)\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t+\left[\left(\frac{1-\alpha_B}{2\beta_B}-1\right)\boldsymbol{M}-\left(1-\frac{\gamma_B}{2\beta_B}\right)\Delta t\boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t\end{aligned} \tag{A.140}$$

A.6.9 Generalized α Method

The generalized α method was introduced by Chung&Hulbert (1993), and considers the following dynamic equilibrium equation:

$$\boldsymbol{M}\ddot{\boldsymbol{U}}_{t+\Delta t-\alpha_m}+\boldsymbol{D}\dot{\boldsymbol{U}}_{t+\Delta t-\alpha_f}+\boldsymbol{K}\boldsymbol{U}_{t+\Delta t-\alpha_f}=\boldsymbol{F}_{t+\Delta t-\alpha_f}(\bar{t})\quad \textit{Generalized } \alpha \textit{ method} \tag{A.141}$$

where

$$\boldsymbol{U}_{t+\Delta t-\alpha_f} = (1-\alpha_f)\boldsymbol{U}_{t+\Delta t} + \alpha_f \boldsymbol{U}_t \tag{A.142}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t-\alpha_f} = (1-\alpha_f)\dot{\boldsymbol{U}}_{t+\Delta t} + \alpha_f \dot{\boldsymbol{U}}_t \tag{A.143}$$

$$\ddot{\boldsymbol{U}}_{t+\Delta t-\alpha_m} = (1-\alpha_m)\ddot{\boldsymbol{U}}_{t+\Delta t} + \alpha_m \ddot{\boldsymbol{U}}_t \tag{A.144}$$

$$\bar{t} = (1-\alpha_f)(t+\Delta t) + \alpha_f t \tag{A.145}$$

with

$$\boldsymbol{U}_{t+\Delta t} = \boldsymbol{U}_t + \Delta t \dot{\boldsymbol{U}}_t + \left(\frac{1}{2} - \beta\right)\Delta t^2 \ddot{\boldsymbol{U}}_t + \beta \Delta t^2 \ddot{\boldsymbol{U}}_{t+\Delta t} \tag{A.146}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \dot{\boldsymbol{U}}_t + (1-\gamma)\Delta t \ddot{\boldsymbol{U}}_t + \gamma \Delta t \ddot{\boldsymbol{U}}_{t+\Delta t} \tag{A.147}$$

Note that the displacement and velocity vectors are the same used for Bossak and HHT methods. Similarly for these methods we can express $\ddot{\boldsymbol{U}}_{t+\Delta t}$ and $\dot{\boldsymbol{U}}_{t+\Delta t}$ as follows:

$$\ddot{\boldsymbol{U}}_{t+\Delta t} = \frac{1}{\beta \Delta t^2}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) - \frac{1}{\beta \Delta t}\dot{\boldsymbol{U}}_t - \left(\frac{1}{2\beta} - 1\right)\ddot{\boldsymbol{U}}_t \tag{A.148}$$

$$\dot{\boldsymbol{U}}_{t+\Delta t} = \frac{\gamma}{\beta \Delta t}\left(\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\boldsymbol{U}}_t + \left(1 - \frac{\gamma}{2\beta}\right)\Delta t \ddot{\boldsymbol{U}}_t \tag{A.149}$$

By substituting (A.142), (A.143), (A.144), and (A.148), (A.149) into the equation (A.141) we can obtain:

$$\boldsymbol{K}^{eff}\boldsymbol{U}_{t+\Delta t} = \boldsymbol{F}^{eff} \tag{A.150}$$

where

$$\begin{aligned}
\boldsymbol{K}^{eff} &= \frac{(1-\alpha_m)}{\beta \Delta t^2}\boldsymbol{M} + \frac{(1-\alpha_f)\gamma}{\beta \Delta t}\boldsymbol{D} + (1-\alpha_f)\boldsymbol{K} \\
\boldsymbol{F}^{eff} &= \boldsymbol{F}_{t+\alpha_f} + \left[\frac{(1-\alpha_m)}{\beta \Delta t^2}\boldsymbol{M} + \frac{(1-\alpha_f)\gamma}{\beta \Delta t}\boldsymbol{D} - \alpha_f \boldsymbol{K}\right]\boldsymbol{U}_t + \\
&+ \left[\frac{(1-\alpha_m)}{\beta \Delta t}\boldsymbol{M} - \left(1 - \frac{\gamma(1-\alpha_f)}{\beta}\right)\boldsymbol{D}\right]\dot{\boldsymbol{U}}_t + \left[\left(\frac{(1-\alpha_m)}{2\beta} - 1\right)\boldsymbol{M} - (1-\alpha_f)\left(1 - \frac{\gamma}{2\beta}\right)\Delta t \boldsymbol{D}\right]\ddot{\boldsymbol{U}}_t
\end{aligned} \tag{A.151}$$

We can verify that when $\alpha_f = \alpha_m = 0$ we fall back into the equations obtained for Newmark's method, (see equations in (A.57)). When $\alpha_f = 0$ and $\alpha_m = \alpha_B$ we fall back into the Bossak's method, (see equations in (A.140)). When $\alpha_m = 0$ we fall back into the HHT method.

A.6.10 Park-Housner's Method

The Park-Housner's method is a semi-implicit method. The mass matrix ($\boldsymbol{M}$) is a diagonal matrix and the stiffness matrix $\boldsymbol{K}$ is split into triangular matrices, i.e. $\boldsymbol{K} = \boldsymbol{K}_L + \boldsymbol{K}_U$, in which $\boldsymbol{K}_L = \boldsymbol{K}_U^T$. The scheme for this method is presented next:

Construction of $\boldsymbol{K}$ and the diagonal mass matrix $\boldsymbol{M}$;

Construction of the matrices $\boldsymbol{K} = \boldsymbol{K}_L + \boldsymbol{K}_U$

Obtain the matrices:

$$\boldsymbol{L} = \boldsymbol{M}(\mathbf{1} + \alpha\beta\Delta t^2 \boldsymbol{M}^{-1}\boldsymbol{K}_L) \quad ; \quad \boldsymbol{Q} = \mathbf{1} + \alpha\beta\Delta t^2 \boldsymbol{M}^{-1}\boldsymbol{K}_U$$
$$\boldsymbol{g}_{t+\Delta t} = \alpha\beta\Delta t^2 \left[\beta \boldsymbol{f}_{t+\Delta t} + (1-\beta)\boldsymbol{f}_t\right] + \boldsymbol{M}(\boldsymbol{U}_t + \beta\Delta t \dot{\boldsymbol{U}}_t)$$

Solve the system:

$$\boldsymbol{L}\boldsymbol{y}_{t+\Delta t} = \boldsymbol{g}_{t+\Delta t} \quad ; \quad \boldsymbol{Q}\boldsymbol{U}^*_{t+\Delta t} = \boldsymbol{y}_{t+\Delta t}$$

Update the variables

$$\boldsymbol{U}_{t+\Delta t} \leftarrow \frac{1}{\beta}\left[\boldsymbol{U}^*_{t+\Delta t} - (1-\beta)\boldsymbol{U}_t\right]$$
$$\dot{\boldsymbol{U}}_{t+\Delta t} \leftarrow \frac{}{\alpha\Delta t}\left[\boldsymbol{U}_{t+\Delta t} - \boldsymbol{U}_t\right] - \frac{(1-\alpha)}{\alpha}\dot{\boldsymbol{U}}_t$$

A.6.11 Trujillo's Method

Trujillo (1977) presented a semi-implicit method, and applied to solve a linear structural dynamic problem. The Trujillo's method separates the stiffness and damping matrices into two upper and lower matrices:

$$\boldsymbol{K} = \boldsymbol{K_L} + \boldsymbol{K_U} \quad ; \quad \boldsymbol{D} = \boldsymbol{D_L} + \boldsymbol{D_U} \tag{A.152}$$

The mass matrix ($\boldsymbol{M}$) is a diagonal matrix.

Trujillo's Method Scheme:

Backward substitution

$$\boldsymbol{U}_{t+\frac{1}{2}} = \boldsymbol{K}^{-1}_{(1)}\left\{\left[\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}_L - \frac{\Delta t^2}{8}\boldsymbol{K}_U\right]\boldsymbol{U}_t + \frac{\Delta t}{2}\left[\boldsymbol{M} + \frac{\Delta t}{4}\left(\boldsymbol{D}_L - \boldsymbol{D}_U\right)\right]\dot{\boldsymbol{U}}_t + \frac{\Delta t^2}{16}\left[\boldsymbol{F}_{t+\Delta t} + \boldsymbol{F}_t\right]\right\}$$

with

$$\boldsymbol{K}_{(1)} = \left[\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}_L + \frac{\Delta t^2}{8}\boldsymbol{K}_L\right]$$

$$\dot{\boldsymbol{U}}_{t+\frac{1}{2}} = \frac{4}{\Delta t}\left[\boldsymbol{U}_{t+\frac{1}{2}} - \boldsymbol{U}_t\right] - \dot{\boldsymbol{U}}_t$$

Forward Substitution

$$\boldsymbol{U}_{t+1} = \boldsymbol{K}^{-1}_{(2)}\left\{\left[\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}_U - \frac{\Delta t^2}{8}\boldsymbol{K}_L\right]\boldsymbol{U}_{t+\frac{1}{2}} + \frac{\Delta t}{2}\left[\boldsymbol{M} + \frac{\Delta t}{4}\left(\boldsymbol{D}_U - \boldsymbol{D}_L\right)\right]\dot{\boldsymbol{U}}_{t+\frac{1}{2}} + \frac{\Delta t^2}{16}\left[\boldsymbol{F}_{t+\Delta t} + \boldsymbol{F}_t\right]\right\}$$

with

$$\boldsymbol{K}_{(2)} = \left[\boldsymbol{M} + \frac{\Delta t}{2}\boldsymbol{D}_U + \frac{\Delta t^2}{8}\boldsymbol{K}_U\right]$$

$$\dot{\boldsymbol{U}}_{t+1} = \frac{4}{\Delta t}\left[\boldsymbol{U}_{t+1} - \boldsymbol{U}_{t+\frac{1}{2}}\right] - \dot{\boldsymbol{U}}_{t+\frac{1}{2}}$$

A.7 Examples

Let us consider the mechanical model with on degree-of-freedom, (see Figure A.12). This mechanical model is made up by a mass body m which is connected by two devices, namely: spring (*Structural*), which is characterized by spring constant k; and by a dashpot with viscosity d (*Damping*), which is responsible for the system energy dissipation. The syste is conservative if $d=0$.

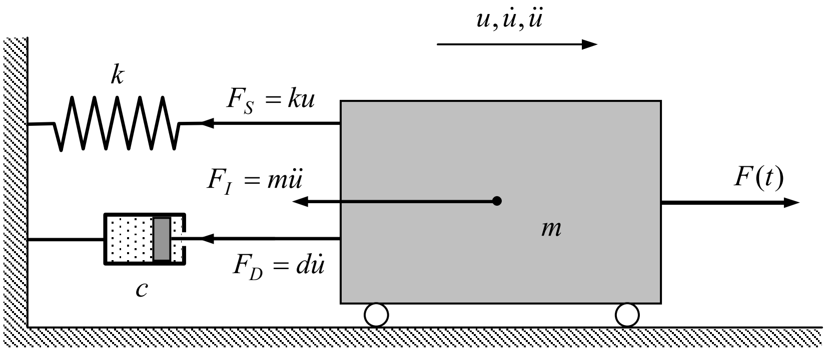

Figure A.12: Mechanical model.

The mechanical problem proposed here has three forces, namely: The inertial force $F_I = m\ddot{u}$, where $\ddot{u} \equiv a$ is the acceleration; F_D is the damping force; and F_S is the spring force associated with the structural stiffness.

The governing equation for this problem, (see Figure A.12), is obtained by force equilibrium:

$$F_I + F_D + F_S = F(t) \tag{A.153}$$

or:

$$\begin{aligned} & m\ddot{u} + d\dot{u} + ku = F(t) \\ & m\frac{d^2u}{dt^2} + d\frac{du}{dt} + ku = F(t) \end{aligned} \tag{A.154}$$

If the body is free of external forces $F(t)=0$, the governing equation is called *free vibration*.

The equation in (A.154) can also be expressed as follows:

$$\begin{aligned} & \ddot{u} + \frac{d}{m}\dot{u} + \frac{k}{m}u = \frac{F(t)}{m} \\ & \ddot{u} + \frac{d}{m}\dot{u} + \omega^2 u = \frac{F(t)}{m} \end{aligned} \tag{A.155}$$

where we have defined the parameter:

$$\omega = \sqrt{\frac{k}{m}} \qquad [\omega] \equiv rad/s \quad \textit{Natural circular frequency} \tag{A.156}$$

A.7.1 Oscillatory Motion

The mechanical model that represents an oscillatory motion is made up by a body mass m connected to the spring with constant k , (see Figure A.13).

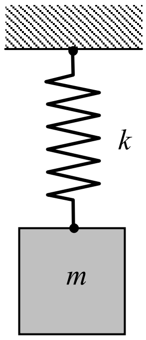

Figure A.13: Mechanical model for an oscillatory problem.

Oscillatory motion is a conservative system, since there is no energy dissipation. The governing equation is represented mathematically by:

$$m\ddot{u} + ku = F = 0 \tag{A.157}$$

with $F = 0$ (free vibration).

Consider as example, $m = 26$, $k = 21000$, and $F = 0$. As boundary and initial conditions we have:

$$u(t = 0) \equiv u_0 = 2 \quad ; \quad \dot{u}(t = 0) \equiv \dot{u}_0 = -3 \tag{A.158}$$

The exact solution is given by the following harmonic function:

$$u(t) = \frac{\dot{u}_0}{\omega}\sin(\omega t) + u_0 \cos(\omega t) \qquad where \qquad \omega = \sqrt{\frac{k}{m}} \tag{A.159}$$

We also define the following parameters for the model:

Natural frequency:

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{\frac{k}{m}}}{2\pi} \approx 2.52\ Hz \quad \textit{Natural frequency} \tag{A.160}$$

Natural period:

$$T = \frac{1}{f} \approx 0.22108\ sec \quad \textit{Natural period} \tag{A.161}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.01$.

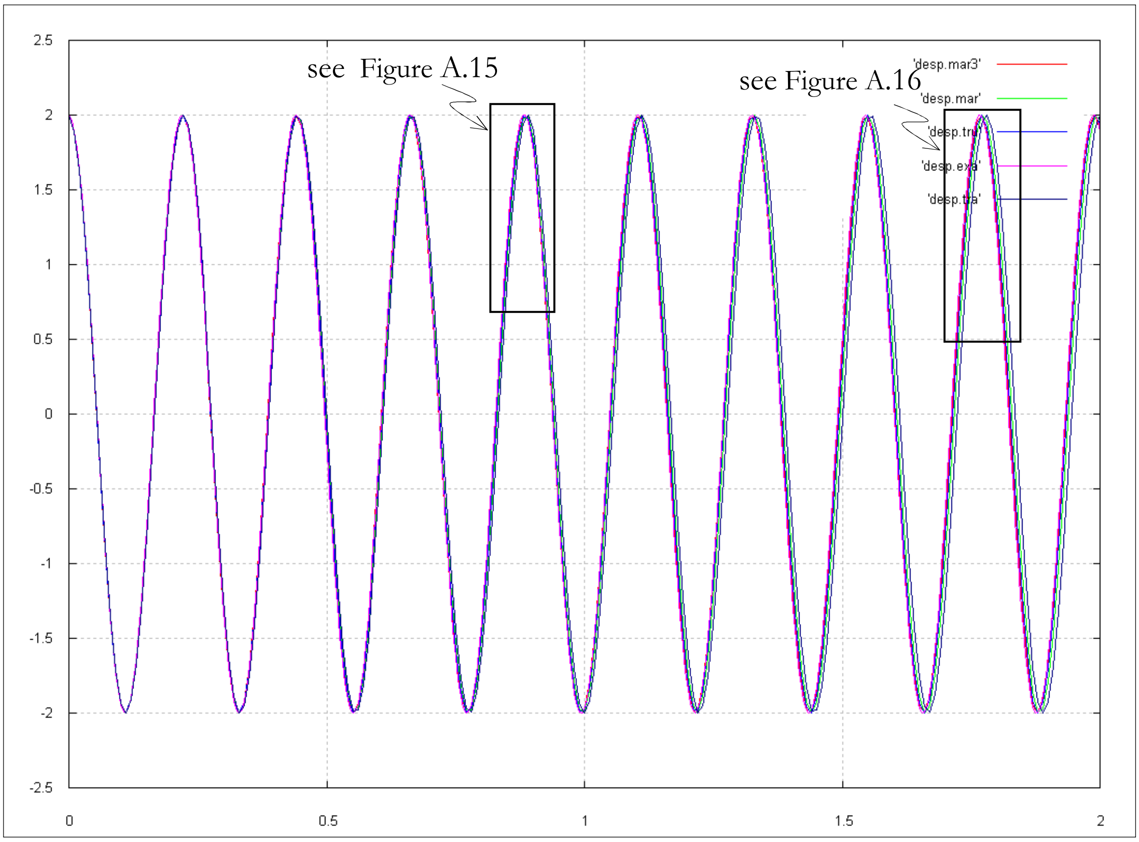

Figure A.14: Displacement vs. time curve ($\Delta t = 0.01$).

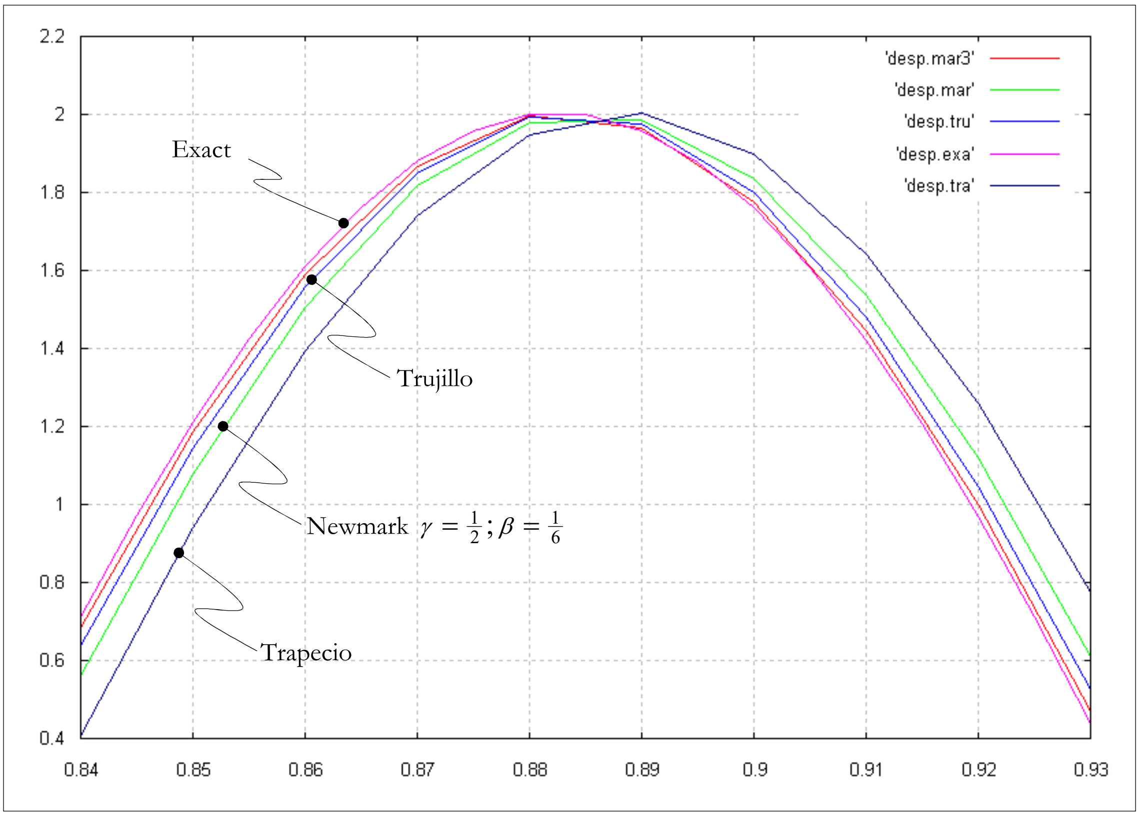

Figure A.15: Displacement vs. time curve[0.84:0.93], ($\Delta t = 0.01$).

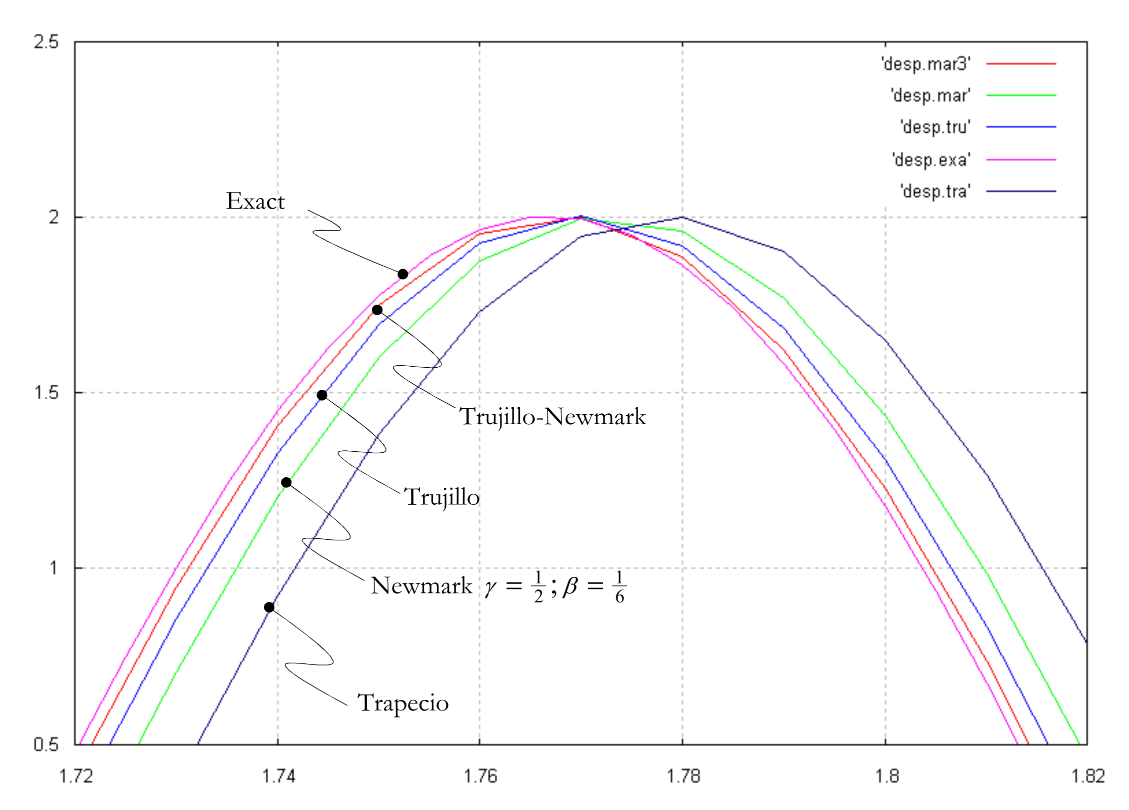

Figure A.16: Displacement vs. time curve [1.72:1.82], ($\Delta t = 0.01$).

A.7.2 Free Vibration with Damping

Let us consider a mechanical model, (see Figure A.17), which represents the free vibration problem ($F(t) = 0$) with damping ($d \neq 0$).

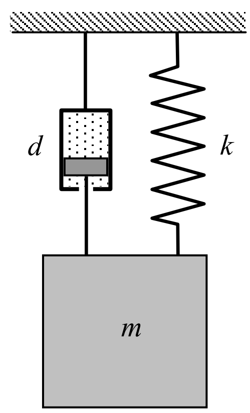

Figure A.17: Mechanical model for free vibration with damping.

As previously seen the governing equation is given by:

$$m\ddot{u} + d\dot{u} + ku = 0 \quad \Rightarrow \quad \ddot{u} + \frac{d}{m}\dot{u} + \frac{k}{m}u = 0 \tag{A.162}$$

By assuming that $u(t) = C_1 \mathrm{e}^{st}$, the above equation becomes:

$$s^2+\frac{d}{m}s+\frac{k}{m}=0 \tag{A.163}$$

whose solutions are given by:

$$s_{1,2}=-\frac{d}{2m}\pm\sqrt{\left(\frac{d}{2m}\right)^2-\frac{k}{m}} \tag{A.164}$$

We could have three possibilities, namely: radicand equals zero (two identical solutions); radicand greater than zero (two different solutions); radicand less than zero (two complex solutions).

- *Radicand equals zero (Critical damping)*

In this case we have:

$$\left(\frac{d}{2m}\right)^2-\frac{k}{m}=0 \quad\Rightarrow\quad \frac{d}{2m}=\sqrt{\frac{k}{m}}=\omega \quad\Rightarrow\quad d=2m\omega \tag{A.165}$$

In this situation the damping coefficient is called *critical damping coefficient*:

$$\boxed{d_c=2m\omega} \quad \textit{critical damping coefficient} \tag{A.166}$$

In general, we define a parameter called damping factor (ζ) which is used to indicate whether the system is *underdamping* ($\zeta<1$, subcritical damping) or *overdamping* ($\zeta>1$, *supercritical damping*). This damping factor is defined by:

$$\boxed{\zeta=\frac{d}{d_c}=\frac{d}{2m\omega}} \quad \textit{Damping factor} \tag{A.167}$$

Note that the equation in (A.163) can be rewritten as follows:

$$s^2+2\omega\zeta s+\omega^2=0 \tag{A.168}$$

whose solutions are given by:

$$s_{1,2}=\left(-\zeta\pm\sqrt{\zeta^2-1}\right)\omega \tag{A.169}$$

For the critical damping case, i.e $\zeta=1$, we have that $s_{1,2}=-\zeta\omega$.

The exact solution for the differential equation (A.162) is given by:

$$u(t)=\exp^{-\omega t}\{[\dot{u}_0+\omega u_0]t+u_0\} \quad \textit{with} \quad (\dot{u}_0\neq 0; u_0\neq 0) \tag{A.170}$$

- *Overcritical damping* $\zeta>1$

In this situation we have two different solutions (and real numbers) given by(A.169). And the solution for the differential equation (A.162) becomes:

$$u(t)=A\exp^{-s_1 t}+B\exp^{-s_2 t} \tag{A.171}$$

where

$$A=\frac{\dot{u}_0+\left(\zeta+\sqrt{\zeta^2-1}\right)\omega u_0}{2\omega\sqrt{\zeta^2-1}} \quad ; \quad B=\frac{-\dot{u}_0-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega u_0}{2\omega\sqrt{\zeta^2-1}} \tag{A.172}$$

- *Subcritical damping* $\zeta < 1$

In this case the solution for the equation (A.169) is given by:

$$s_{1,2} = \left(-\zeta \pm i\sqrt{1-\zeta^2} \right)\omega \tag{A.173}$$

And the solution for the differential equation (A.162) becomes:

$$u(t) = \exp^{-\zeta\omega t}\left[A\sin\left(\omega t\sqrt{1-\zeta^2} \right) + B\cos\left(\omega t\sqrt{1-\zeta^2} \right) \right] \tag{A.174}$$

or:

$$u(t) = \exp^{-\zeta\omega t}\left[\frac{\dot{u}_0 + \zeta\omega u_0}{\omega_d}\sin(\omega_d t) + u_0\cos(\omega_d t) \right] \tag{A.175}$$

where $\omega_d = \omega\sqrt{1-\zeta^2}$

A.7.2.1 Free Vibration with Damping Example

As example, consider that $m = 0.0052$, $d = 0.1$, $k = 12$, and boundary and initial conditions:

$$u_0 = 1.5 \quad ; \quad \dot{u}_0 = 0 \tag{A.176}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.017$.

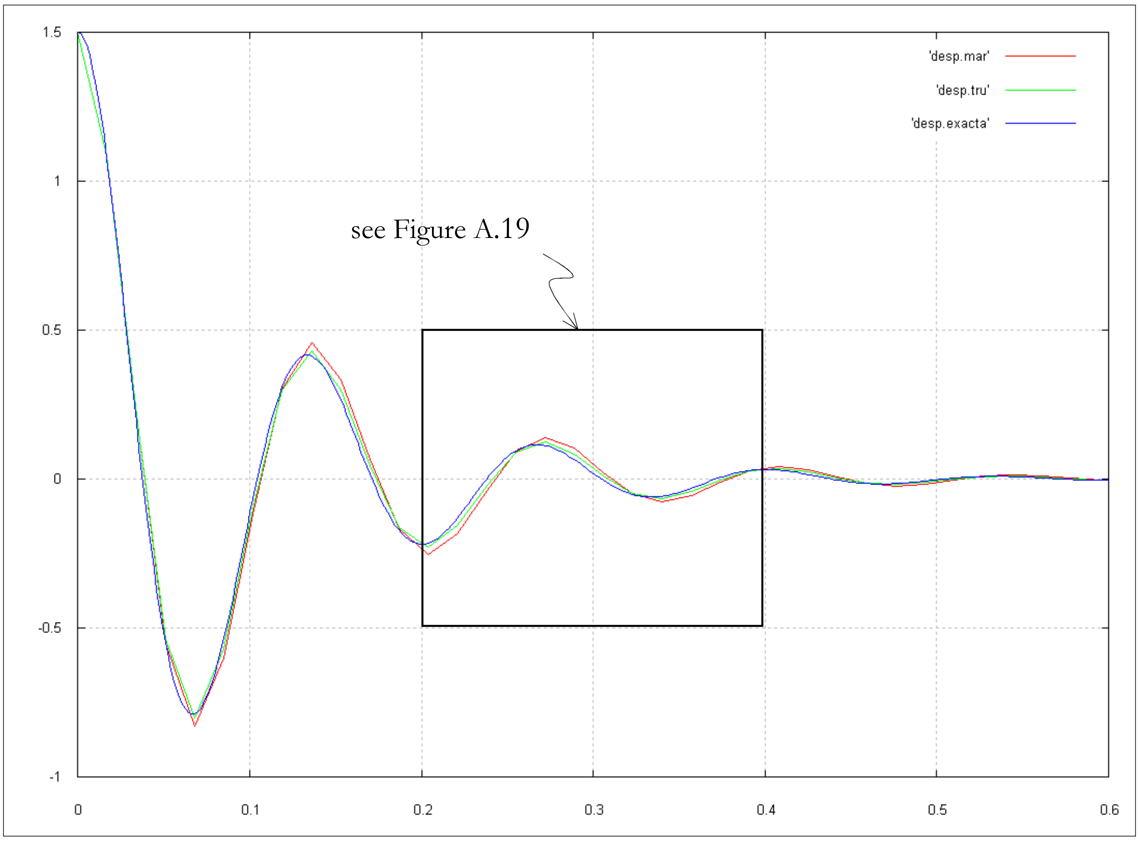

Figure A.18: Displacement vs. time curve, ($\Delta t = 0.017$).

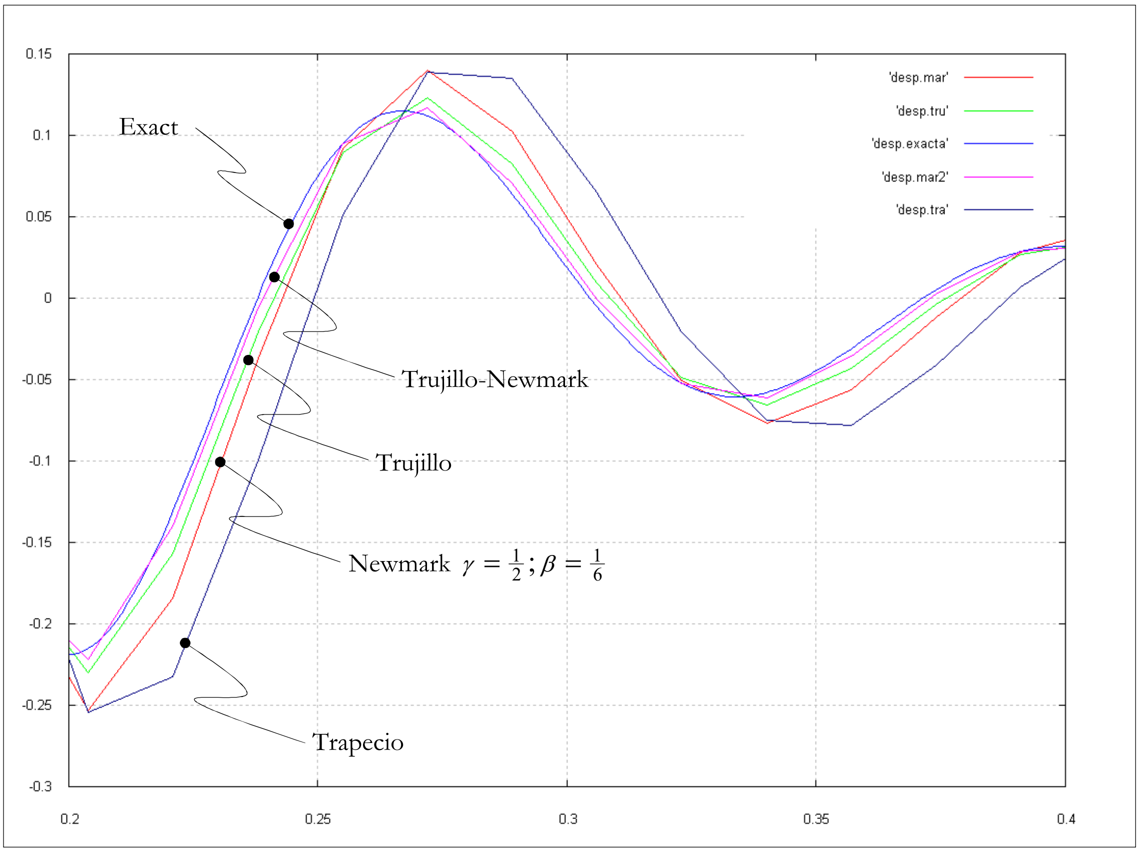

Figure A.19: Displacement vs. time curve [0.2:0.4], ($\Delta t = 0.017$).

A.7.3 Miscellaneous Examples

A.7.3.1 Pendulum

Let us consider the pendulum problem, (see Figure A.20).

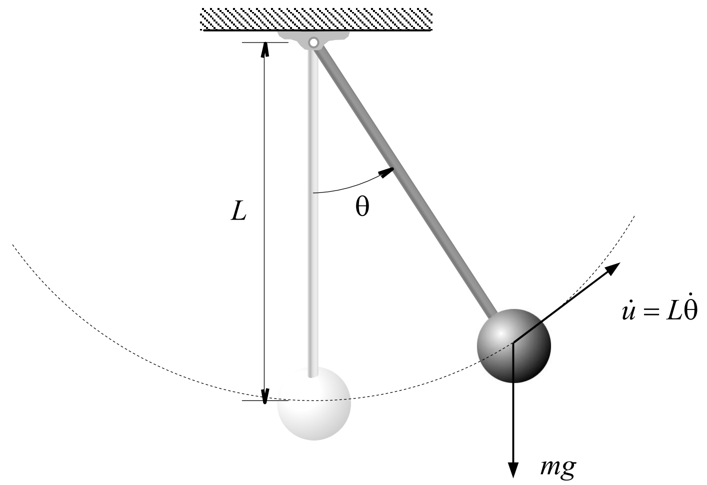

Figure A.20: Pendulum.

The governing equation for this problem is given by:

$$mL^2\ddot{\theta} + mgL\sin(\theta) + dL\dot{\theta} = 0 \tag{A.177}$$

The above equation can be rewritten as follows:

$$m\ddot{\theta} + \frac{d}{L}\dot{\theta} = -\frac{mg}{L}\sin(\theta) = F(t,\theta) \tag{A.178}$$

We consider the parameter values: $L = 10.0$, $m = 1.0$, $d = 2.0$, $g = 10$, and boundary and initial conditions:

$$\theta(t=0) = 0 \quad ; \quad \dot{\theta}(t=0) = 3 \tag{A.179}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.05$.

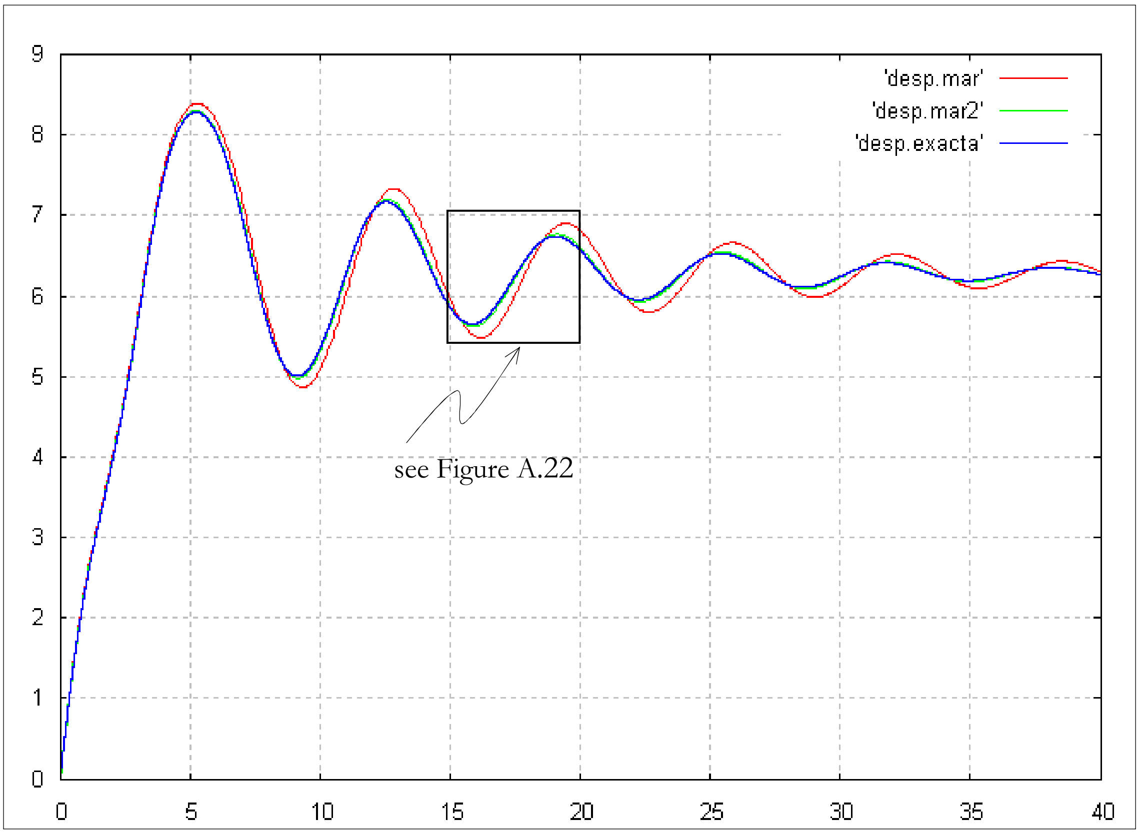

Figure A.21: Displacement vs. time curve, ($\Delta t = 0.05$).

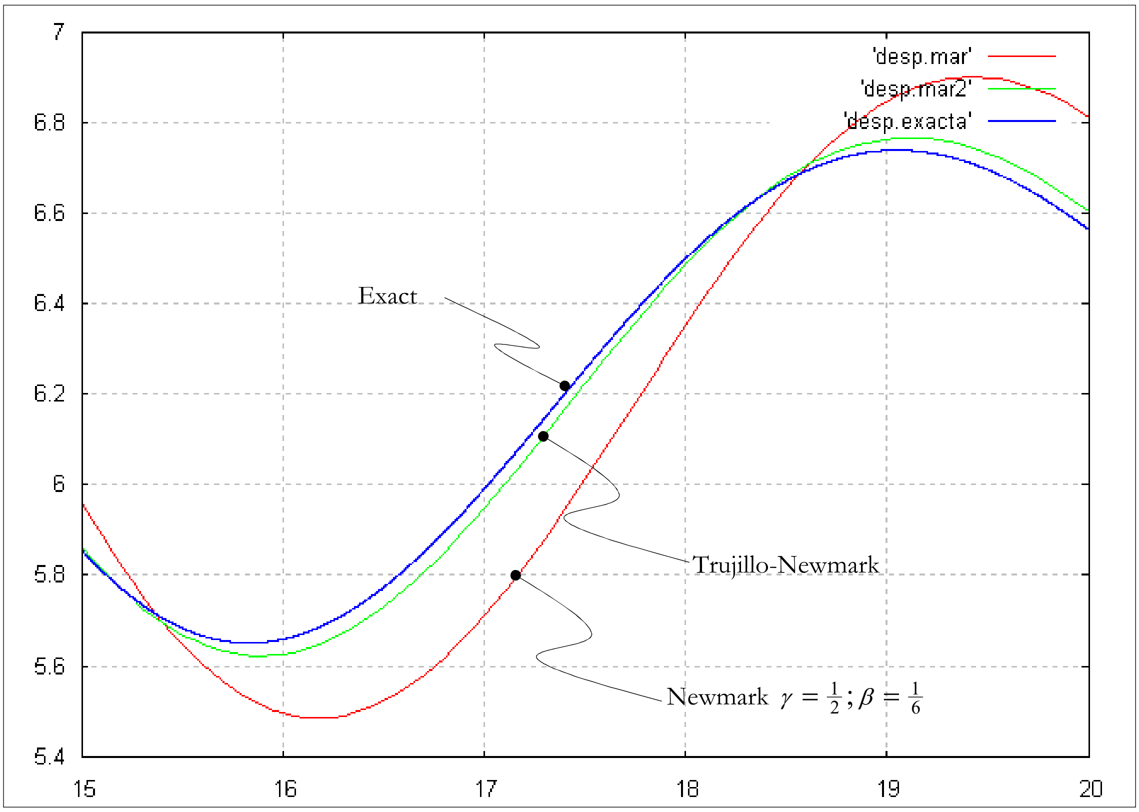

Figure A.22: Displacement vs. time curve [15:20], ($\Delta t = 0.05$).

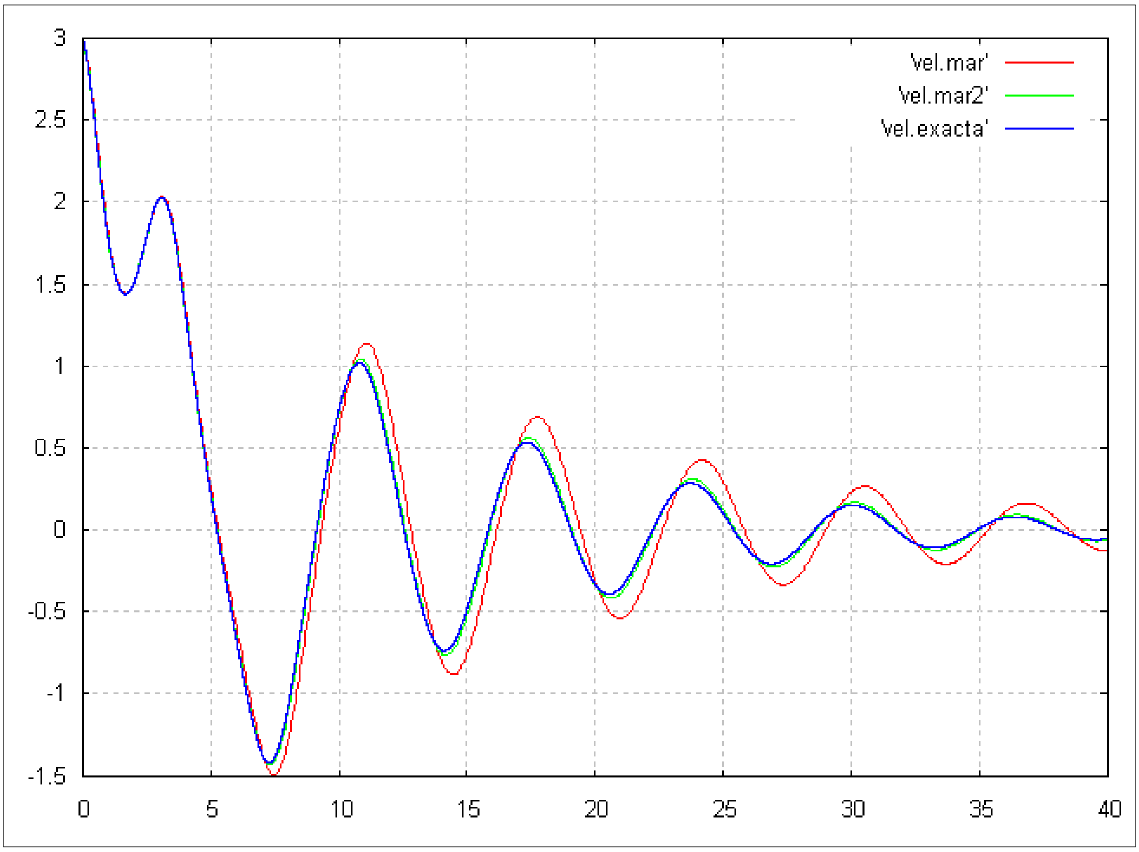

Figure A.23: Velocity vs. time curve, $(\Delta t = 0.05)$.

A.7.3.2 Dynamics of the van der Pol Equation

Let us consider the differential equation:

$$\ddot{\nu} + \alpha(\nu^2 - 1)\dot{\nu} + \omega^2 \nu = 0 \qquad \text{(A.180)}$$

which is known as van der Pol oscillator. The above equation can be rewritten as follows:

$$\ddot{\nu} - \alpha\dot{\nu} + \omega^2 \nu = -\alpha \nu^2 \dot{\nu} = F(t, \nu, \dot{\nu}) \qquad \text{(A.181)}$$

By considering the dynamic equilibrium equation:

$$m\ddot{u} + d\dot{u} + ku = F \quad \Rightarrow \quad \ddot{u} + \frac{d}{m}\dot{u} + \frac{k}{m}u = \frac{F}{m} \qquad \text{(A.182)}$$

and by comparing the equations (A.181) and (A.182) we can conclude that: $\frac{d}{m} = -\alpha$, $\frac{k}{m} = \omega^2$, $\frac{F(t,\nu,\dot{\nu})}{m} = -\alpha\nu^2\dot{\nu}$. Consider as example the values: $m = 1$, $d = -\alpha = -8$, $k = \omega^2 = 0.25$, and boundary and initial conditions:

$$\nu(t=0) \equiv \nu_0 = 1 \quad ; \quad \dot{\nu}(t=0) \equiv \dot{\nu}_0 = 3 \qquad \text{(A.183)}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.01$.

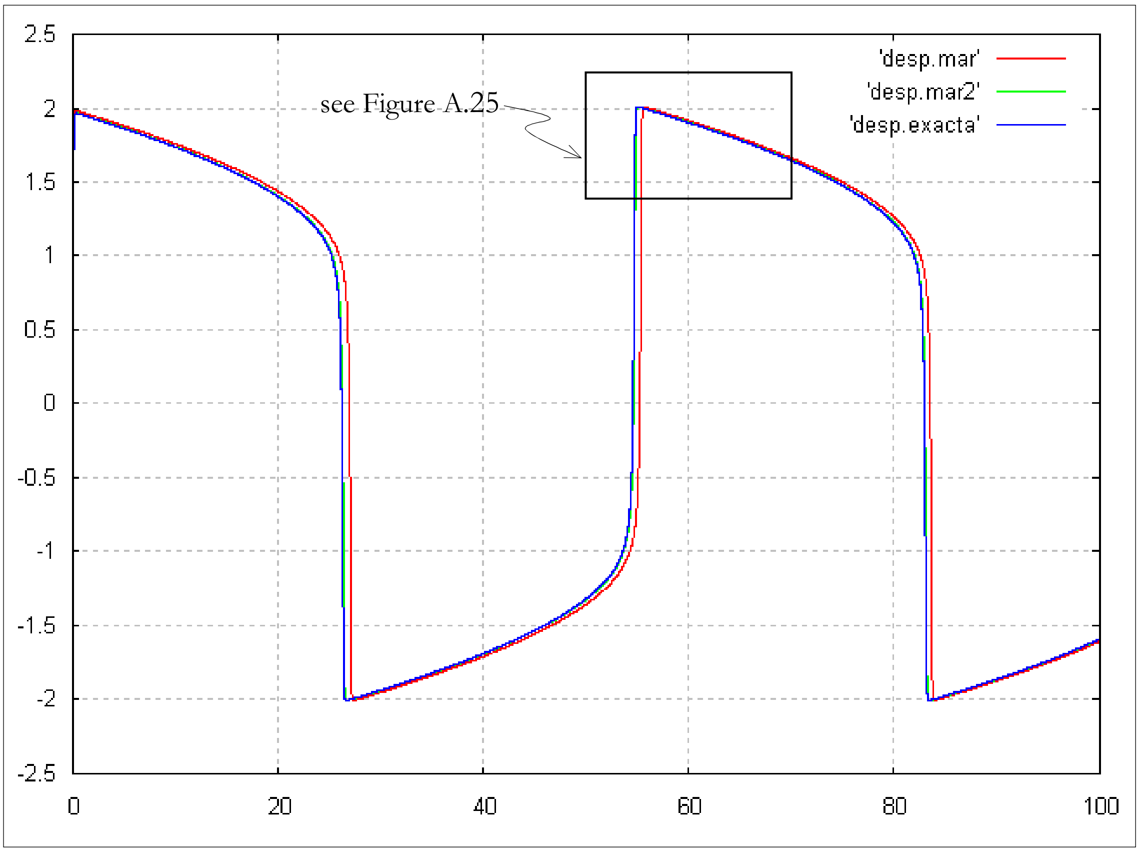

Figure A.24: "Displacement" vs. time curve, ($\Delta t = 0.01$).

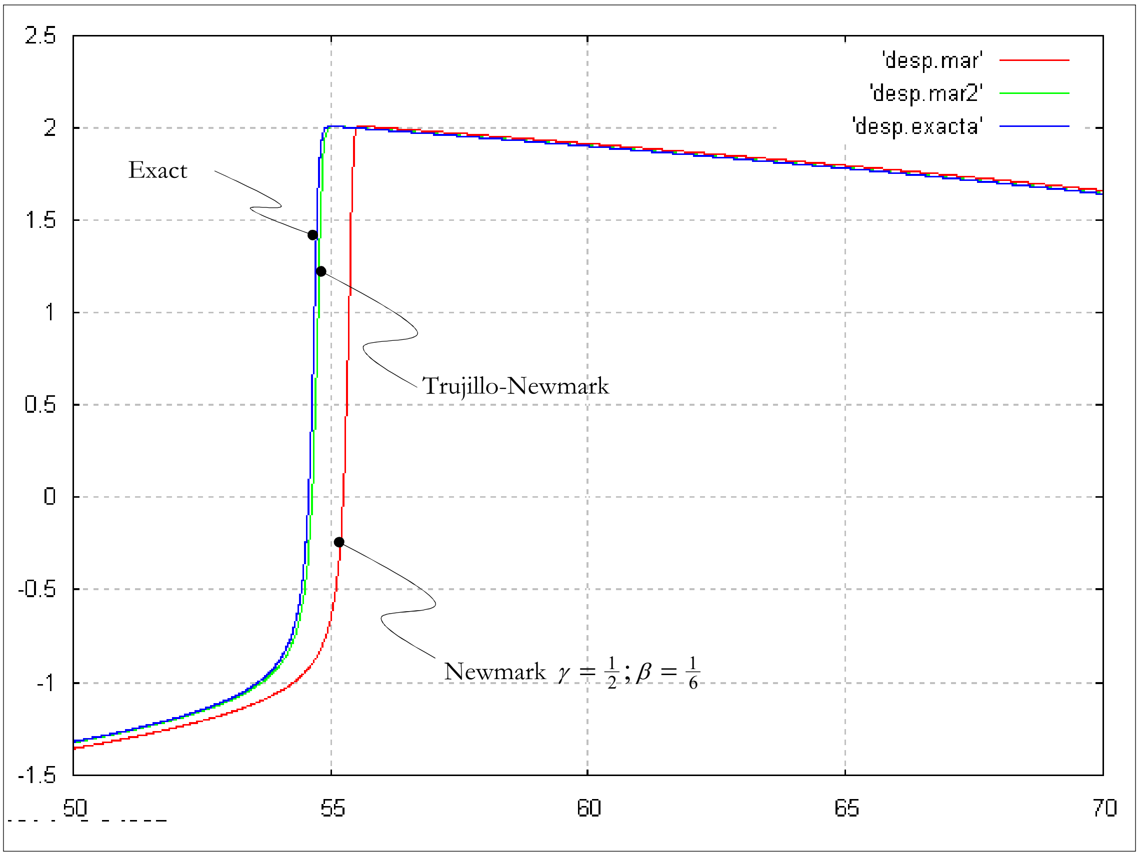

Figure A.25: "Displacement" vs. time curve [50:70], ($\Delta t = 0.01$).

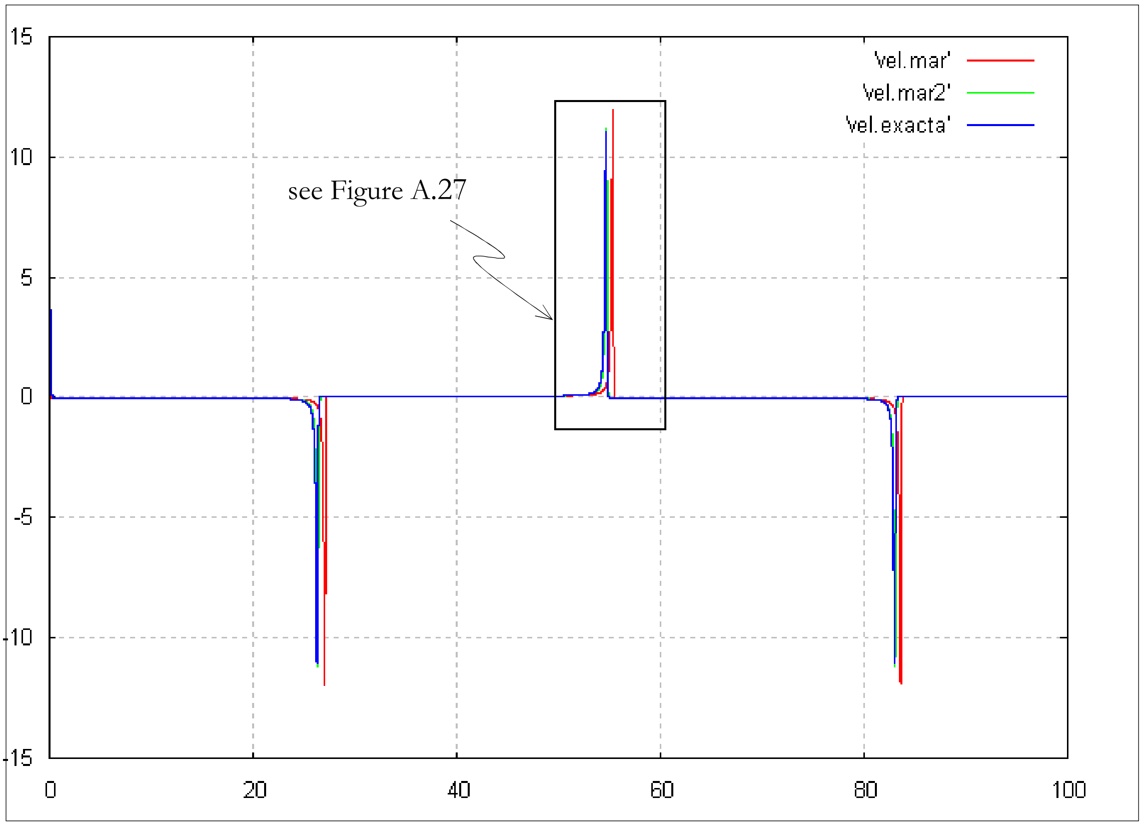

Figure A.26: “Velocity” vs. time curve, ($\Delta t = 0.01$).

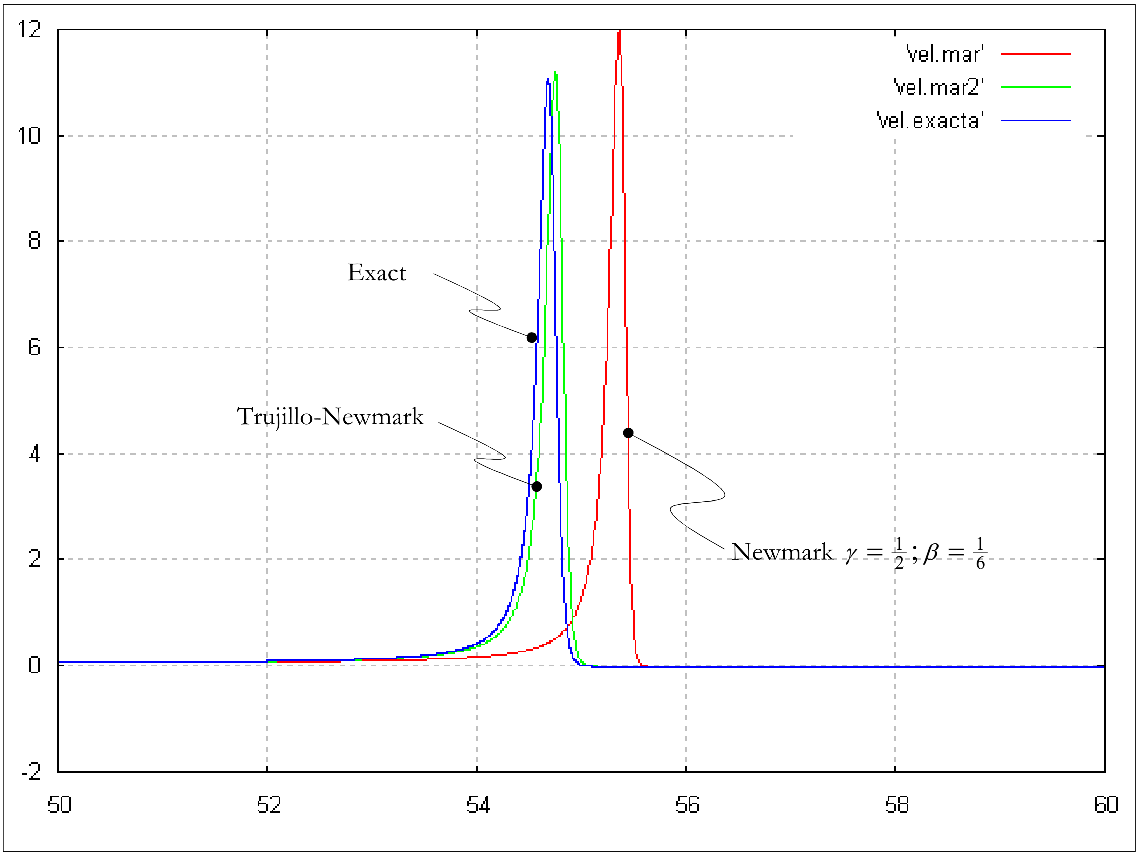

Figure A.27: “Velocity” vs. time [50:60], ($\Delta t = 0.01$).

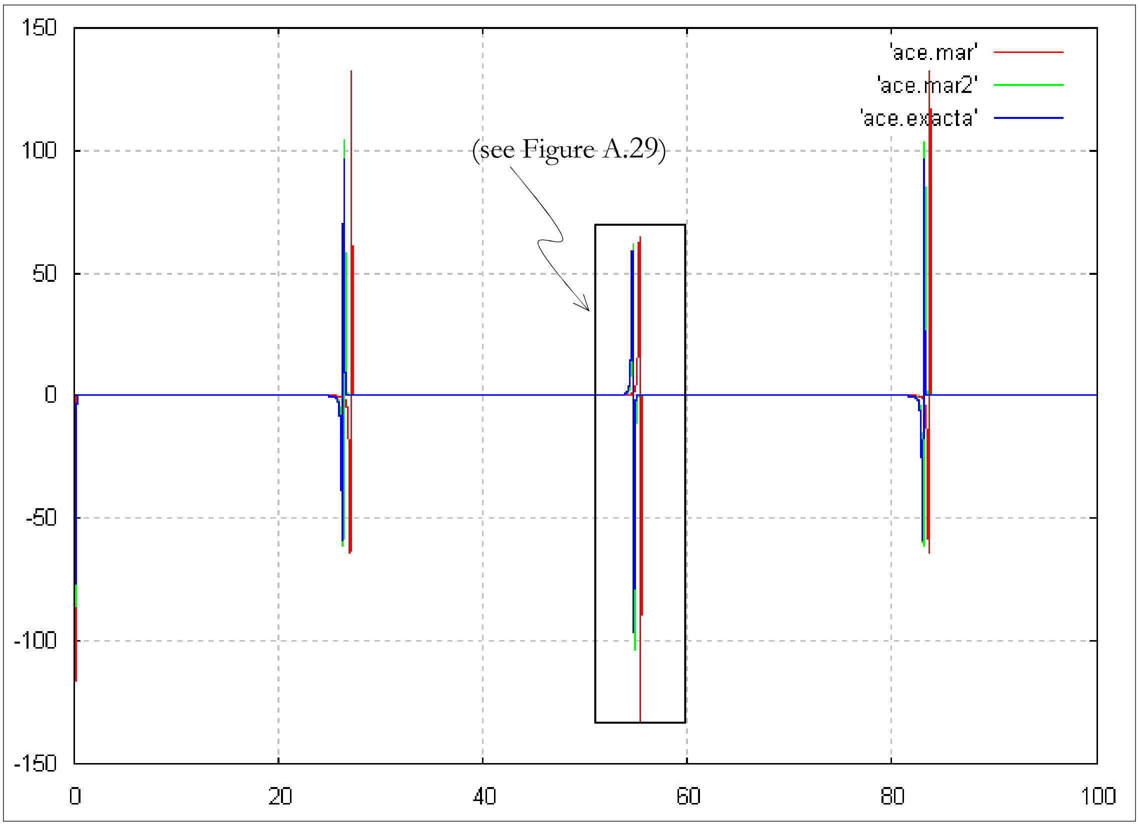

Figure A.28: "Acceleration" vs. time curve, ($\Delta t = 0.01$).

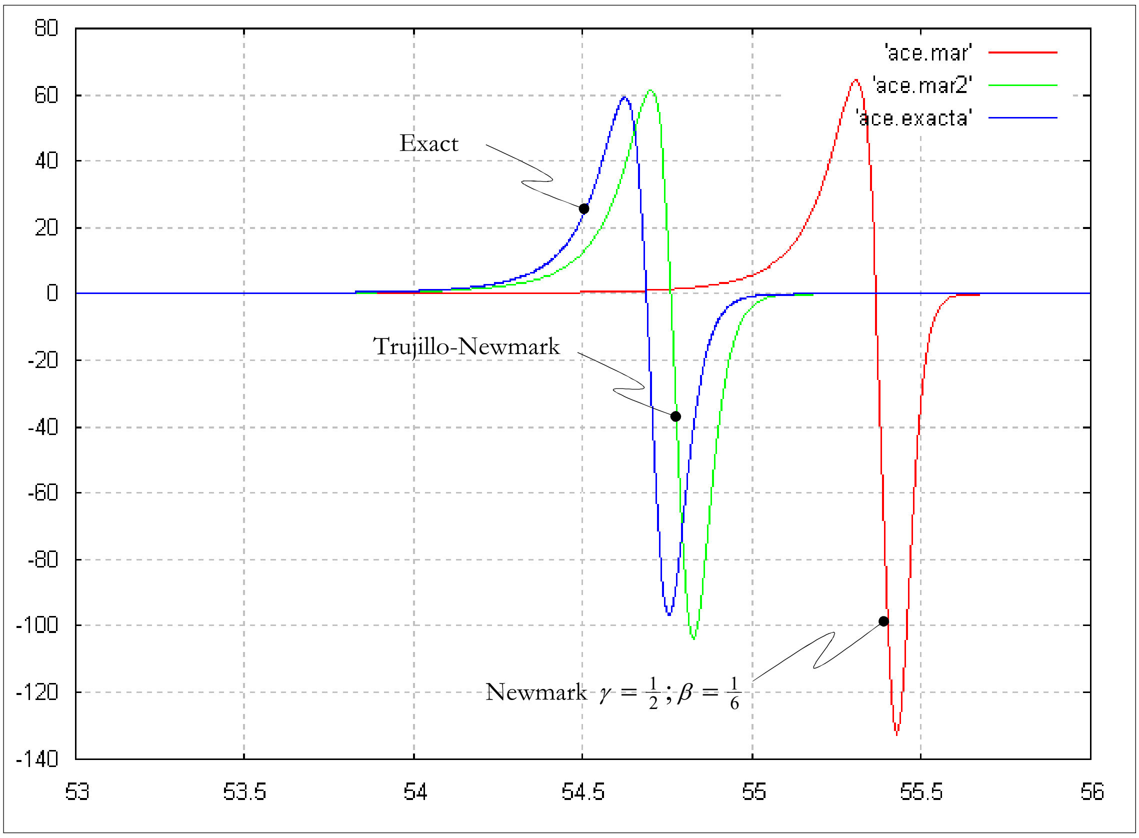

Figure A.29: "Acceleration" vs. time curve [53:56], ($\Delta t = 0.01$).

Next we present the results by using the direct integration with time increment equal to $\Delta t = 0.03$.

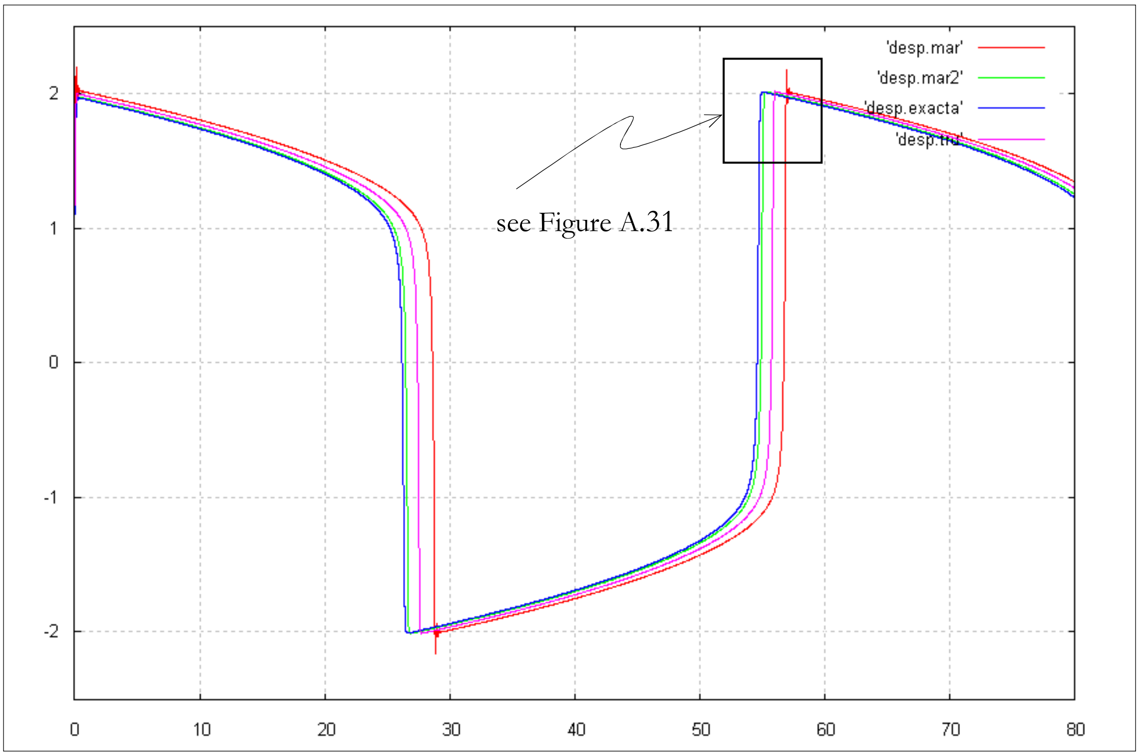

Figure A.30: "Displacement" vs. time curve, ($\Delta t = 0.03$).

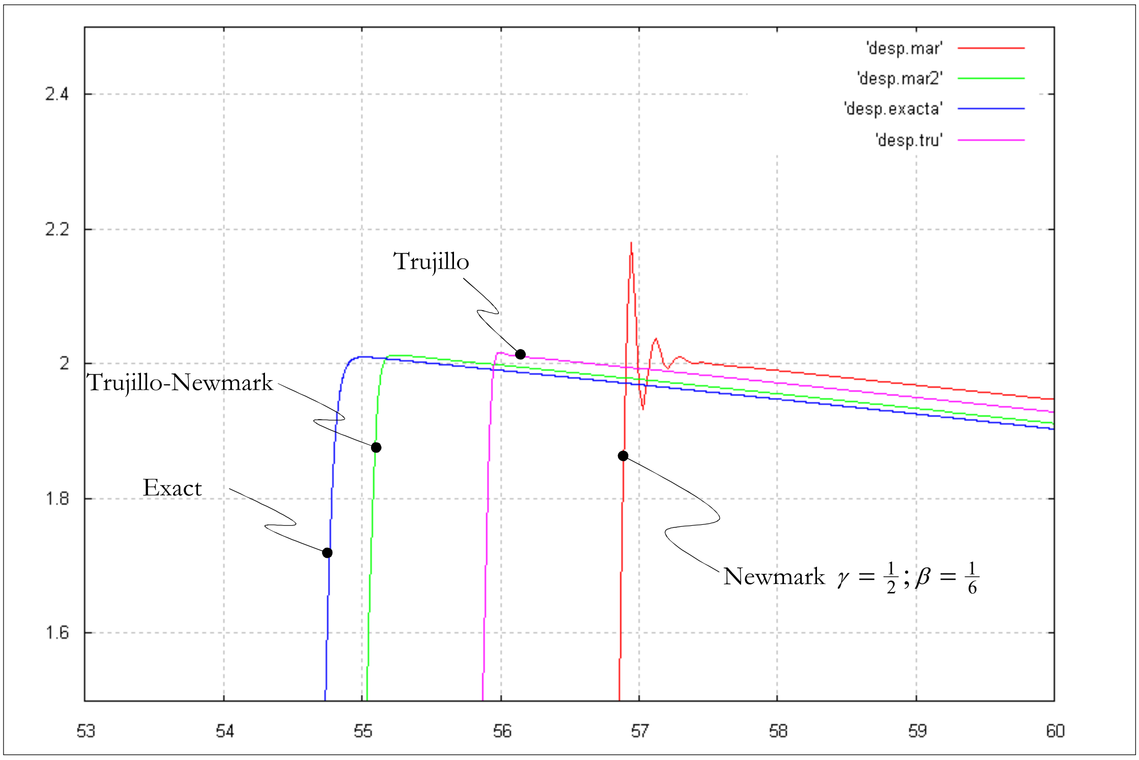

Figure A.31: "Displacement" vs. time curve [53:60], ($\Delta t = 0.03$).

By means of numerical integration we present the results using the time increment $\Delta t = 0.05$.

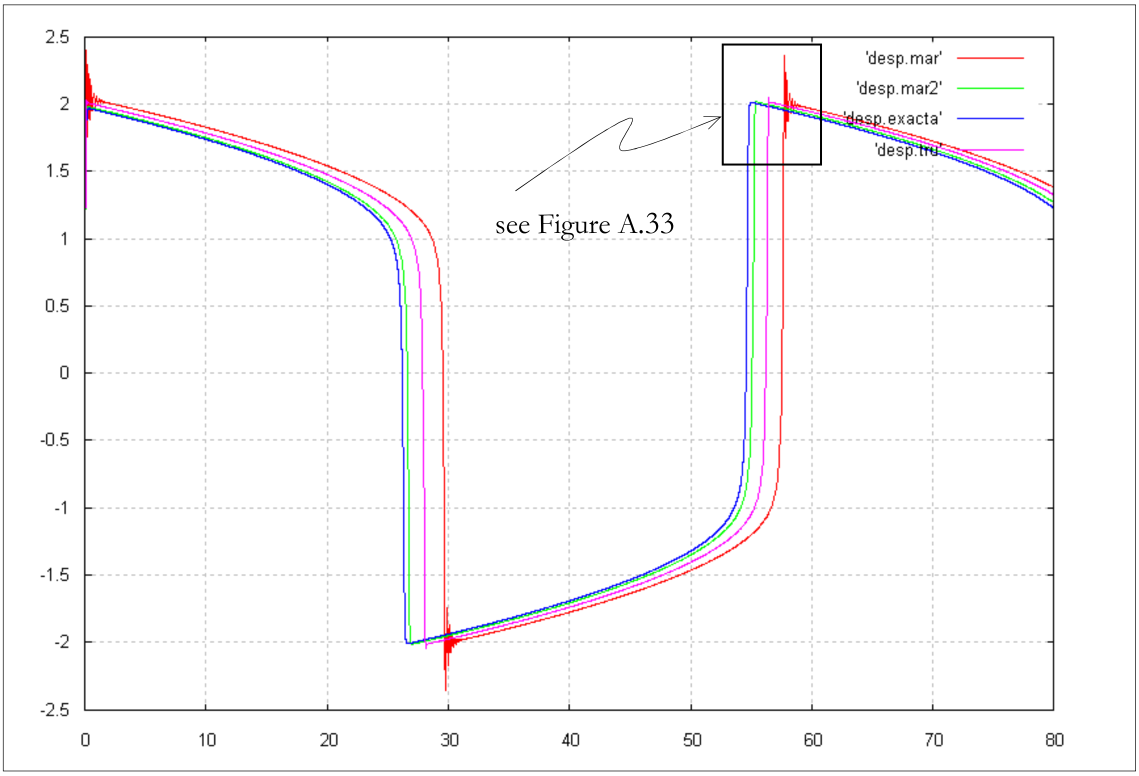

Figure A.32: "Displacement" vs. time curve, ($\Delta t = 0.05$).

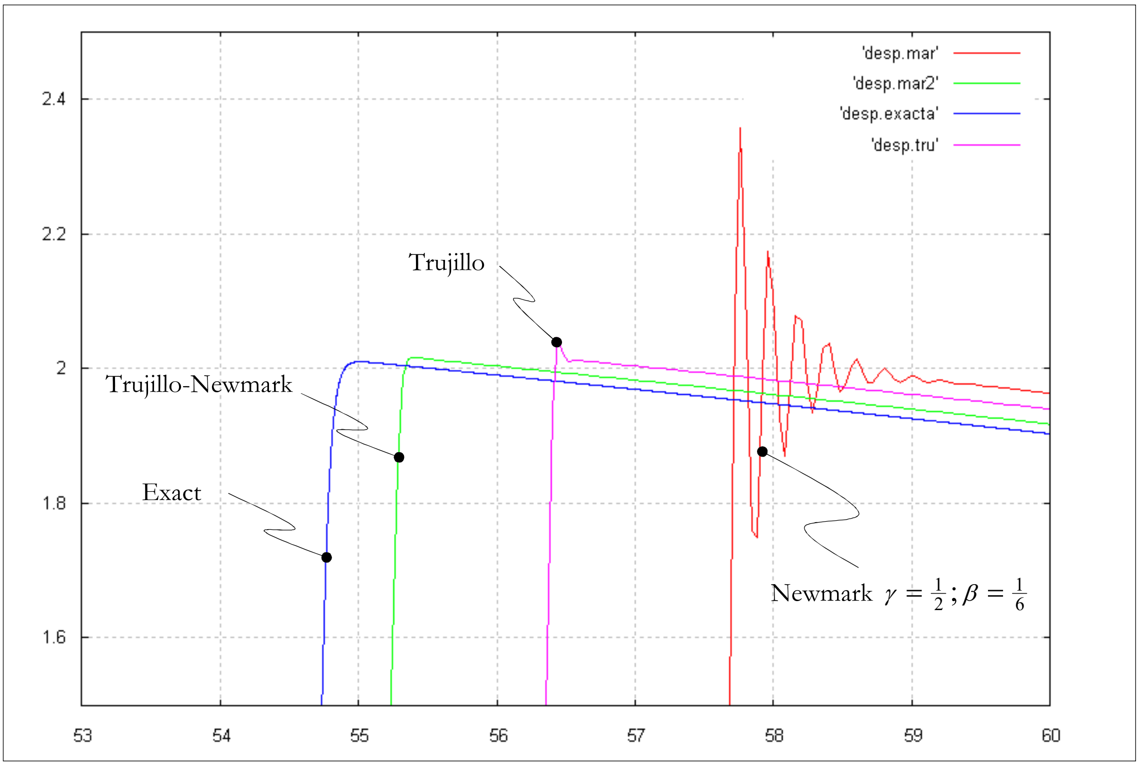

Figure A.33: "Displacement" vs. time curve [53:60], ($\Delta t = 0.05$).

In Figure A.34 we show the “displacement” vs. time curve for different values for α .

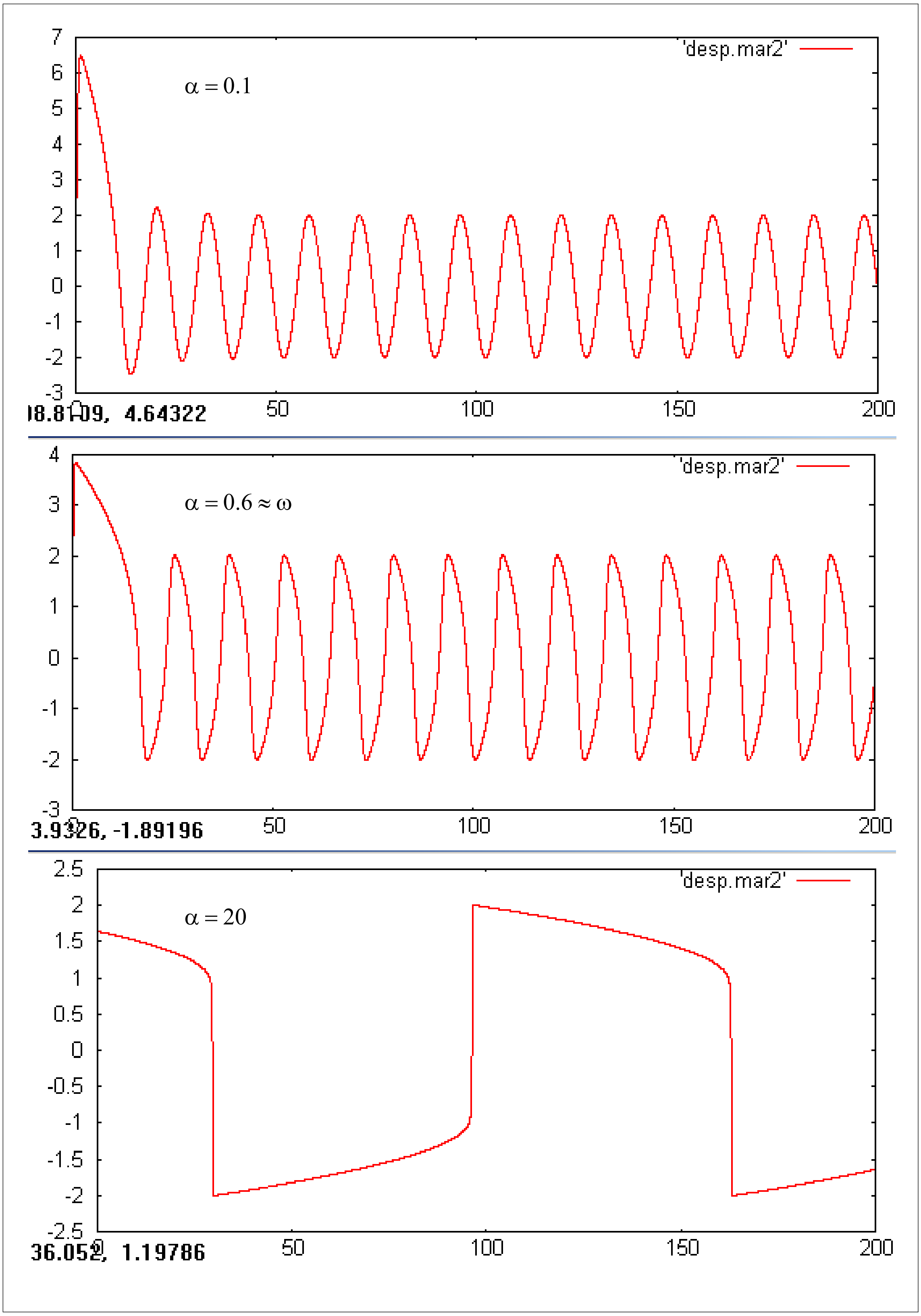

Figure A.34: “Displacement” vs. time curve.

A.7.3.3 Forced Harmonic Response without Damping

Consider the differential equation:

$$m\ddot{u} + ku = F_0 \sin(\Omega t) \tag{A.184}$$

where Ω is the excitation frequency. When $\Omega = \omega$, resonance phenomena appears.

As example consider that: $m = 4.5$, $k = 3500$, $F_0 = 100$, $\Omega = 18$, and boundary and initial conditions:

$$u(t=0) = 15 \quad ; \quad \dot{u}(t=0) = 150 \tag{A.185}$$

By means of numerical integration we present the results using the time increment $\Delta t = 0.01$.

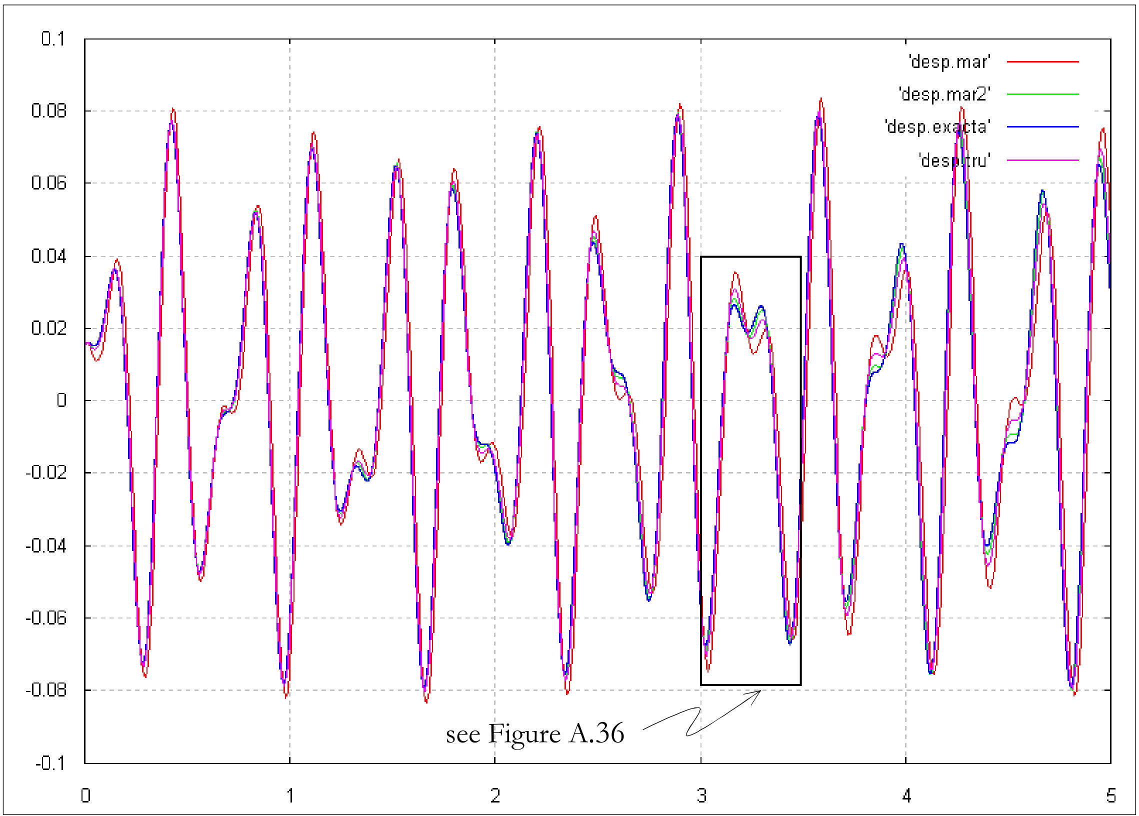

Figure A.35: Displacement vs. time curve, ($\Delta t = 0.01$).

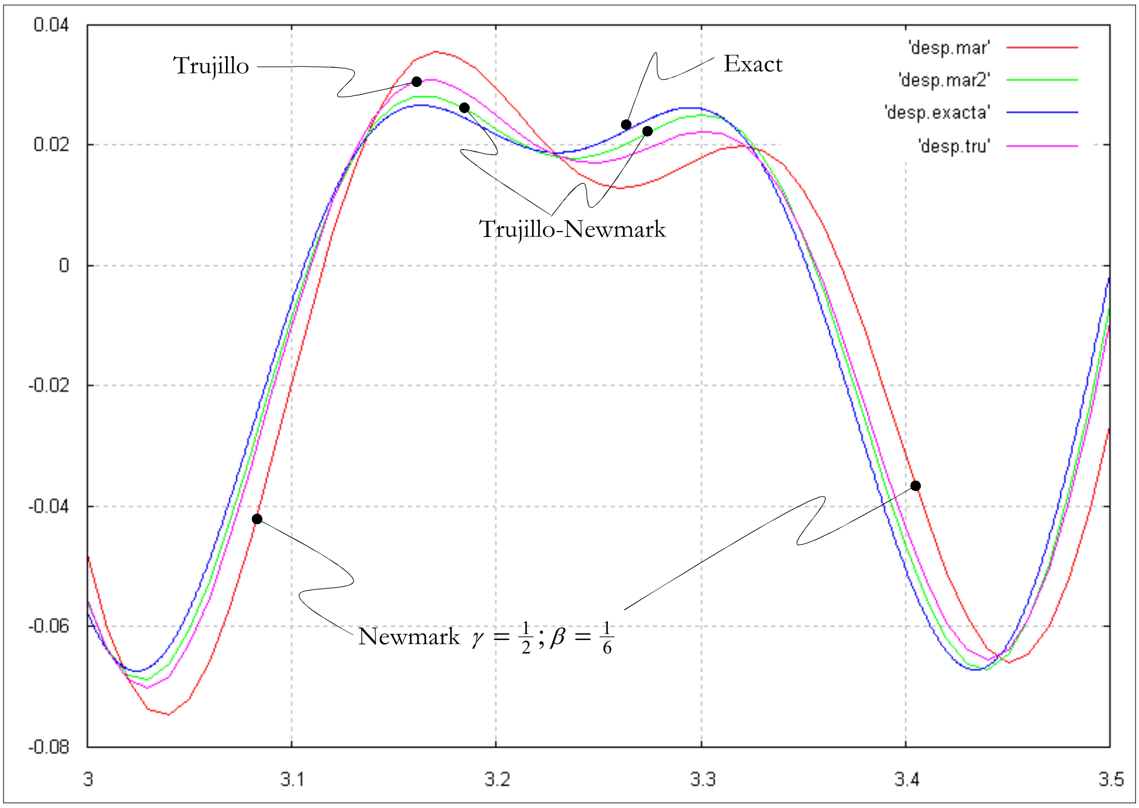

Figure A.36: Displacement vs. time curve [3:3.5], ($\Delta t = 0.01$).

Now consider that:

$$u(t=0)=0 \quad ; \quad \dot{u}(t=0)=0 \tag{A.186}$$

besides we consider that $\Omega = \omega$. With these conditions we can note that the system enter in resonance, (see Figure A.37).

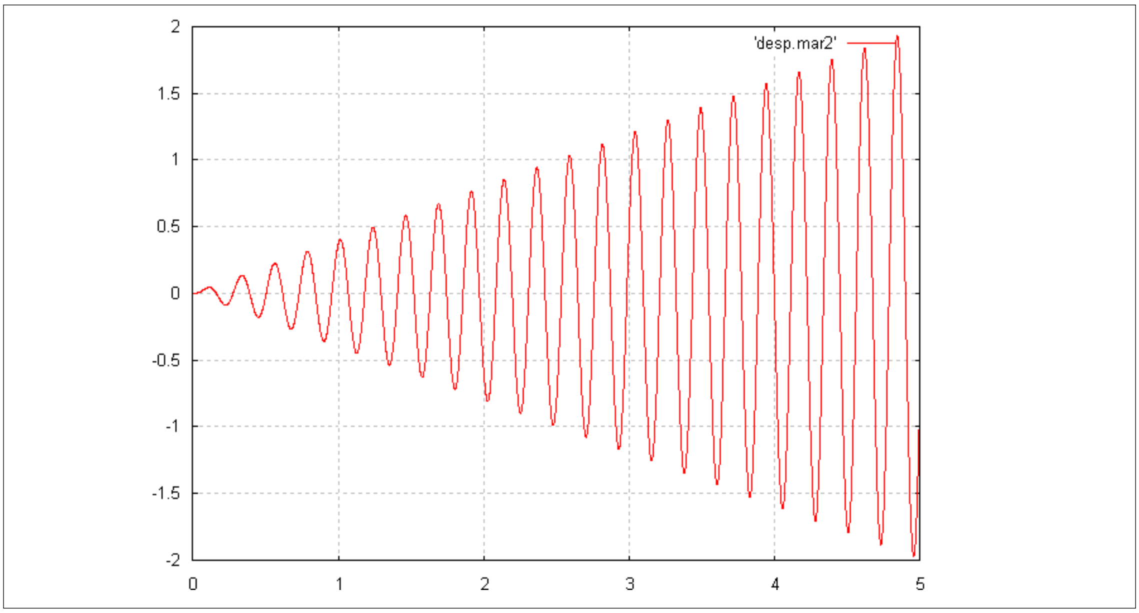

Figure A.37: Displacement vs. time curve.

Dynamics Structures References

CHAPRA, S.C. & CANALE, R.P. (1988). *Numerical Method for engineering.* McGraw-Hill.

CHUNG. J. & HULBERT (1993). A time integration algorithm for structural dynamics with improved numerical dissipation: The Generalized-α method. *Journal of Applied Mechanics.* Vol. 60. pp. 371-375.

COOK, R.D.; MALKUS, D.S. & PLESHA, M.E. (1989). *Concepts and applications of finite element analysis.* John Wiley & Sons.

CRISFIELD, M.A.(1997). *Non-linear finite element analysis of solids and structures.* Vol.2. John Wiley & Sons, England.

HILBER, H.; HUGHES, T.J.R. & TAYLOR, R. L. (1977). Improved numerical dissipation for time integration algorithms in structural dynamics. *Earthquake Engineering and Structural Dynamics.* Nº5, pp. 283-292.

PARK, K. & HOUSNER, J.M. (1982). Semi-implicit transient analysis procedures for structural dynamics analysis. *Int. J. Num. Meth. Eng.* Vol. 18, pp. 609-622.

TEDESCO, J.M.; MCDOUGAL, W.G. & ROSS, C.A.(1998). *Structural dynamics: theory and applications.* Addison Wesley Longman, Inc.

TRUJILLO, D.M. (1977). An unconditionally stable explicit algorithm for structural dynamics. *Int. J. Num. Meth. Eng.* Vol. 11. pp. 1579-1592.

WOOD, W.L.; BOSSAK, M. & ZIENKIEWICZ, O.C. (1981). An alpha modification of Newmark's methods. *Int. J. Num. Meth. Eng.* Vol. 15. pp. 1562-1566.